Selected Approaches to Estimation of Signal Phase

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Abstract

A number of approaches to parametric estimation of signal phase are addressed. The goal is to estimate the phase, or more precisely, the parameters of the phase of a signal corrupted by white Gaussian noise. Signal models are restricted to those with linear and quadratic phase, but many of the results presented are applicable to signals with higher order phase laws. Results highlight the classic trade-off between estimator complexity and estimator variance.

1 Introduction

This paper addresses the estimation of the phase of a signal corrupted by additive white Gaussian noise. Estimation of signal phase is necessary in a variety of applications. The phase of a frequency modulated (FM) radio signal needs to be accurately estimated in order to recover the message signal. In a military environment, estimates of the initial phase, phase rate (frequency or Doppler), and phase acceleration (frequency rate or Doppler rate) are all used to provide information about a target. Estimation of the frequency (phase rate of change) of a sinusoid is another common estimation problem, arising in a number of areas.

Techniques for estimation of the phase of a signal fall into two broad categories: parametric and non-parametric. The parametric model nearly always employs a polynomial phase law [1]. Such a model is used when the phase law is know, as is the case with a sinusoid (linear phase) or a chirp (quadratic phase), or when one wishes to approximate a signal with unknown phase law by a signal with a polynomial phase law. In the latter case, successive approximations of increasing order might be made until reduction of modeling error plateaus. If a polynomial phase is not a good fit for the problem, as might be the case with FM radio for example, the signal phase can be *tracked* using one of a variety of non-parametric phase estimation techniques. Techniques for phase tracking include phase locked loops (PLLs), adaptive least mean squares (LMS), and moments of time-frequency distribution (TFDs). All phase tracking methods exhibit a trade-off between averaging (for robustness to noise) and ability to response to rapid phase changes. Alternatively, a signal's phase can be modeled locally by a polynomial phase. A sliding window can be applied, a local estimate determined, the window moved, and the process repeated.

In this paper we will assume that the signal phase obeys a polynomial phase law. The entire data record length will be used to estimate the phase parameters. Two signal models will be examined: 1.) A constant frequency (linear phase) signal, and

2.) a chirp signal (quadratic phase law). We will first give the maximum likelihood (ML) solution to the two problems, and then investigate an alternate approach using an approximation that is accurate at high SNRs. First, however, we will define a signal's phase and illustrate the problems inherent in its computation.

2 Estimation of Signal Phase

Let g_n be a sequence of complex values defined by

$$g_n = A_n \mathrm{e}^{j\Phi_n} \qquad n = n_0,\ n_0+1,\ \ldots,\ n_0+N-1 \tag{1}$$

The phase is defined as the argument of the sequence.

$$\Phi_n = \arg(g_n) \tag{2}$$

Although we have defined our data to be a complex sequence, data in the real world is most commonly real valued. Computation of the phase of a real sequence is a difficult task, however. Real data is therefore transformed to a complex sequence (having the same phase) using an operation known as the Hilbert transform (or by complex demodulation). Given a real sequence, s_n, the *analytic* signal is defined as

$$x_n = s_n + j\mathrm{H}[s_n] \tag{3}$$

where H[ ] is the discrete-time version of the Hilbert transform operation defined in [2]. This operation will map the real sequence $s_n = A_n \cos(\Phi_n)$ to the complex analytic sequence $x_n = A_n \mathrm{e}^{j\Phi_n}$ under many conditions.

Although complex data sequences are often derived from real data sequences as described above, we make no such assumptions here. That is, we will assume that we have been given a complex data sequence generated according to Equation (1). With that assumption, the subsequent sections will describe the task of determining Φ_n.

2.1 General Phase Unwrapping

Determination of Φ_n requires the computation of the inverse tangent

$$\hat{\Phi}_n = \tan^{-1}\left(\frac{\mathrm{Im}\{g_n\}}{\mathrm{Re}\{g_n\}}\right) \tag{4}$$

$\hat{\Phi}_n$ will only equal Φ_n when Φ_n lies in the interval $(-\pi,\ \pi)$. Values of Φ_n with magnitude greater than π will be mapped back to this interval by the inverse tangent, which is a many-to-one function. For example let $A_n = 1$ and $\Phi_n = n$. Equation (1) then becomes

$$g_n = \mathrm{e}^{jn} \tag{5}$$

The computed phase $\hat{\Phi}_n$ is plotted along side the actual phase Φ_n in Figure 1. Note that when Φ_n exceeds π in magnitude, it is mapped back into the interval $(-\pi,\ \pi)$. This is the so-called *wrapping* of the phase function. The standard technique for

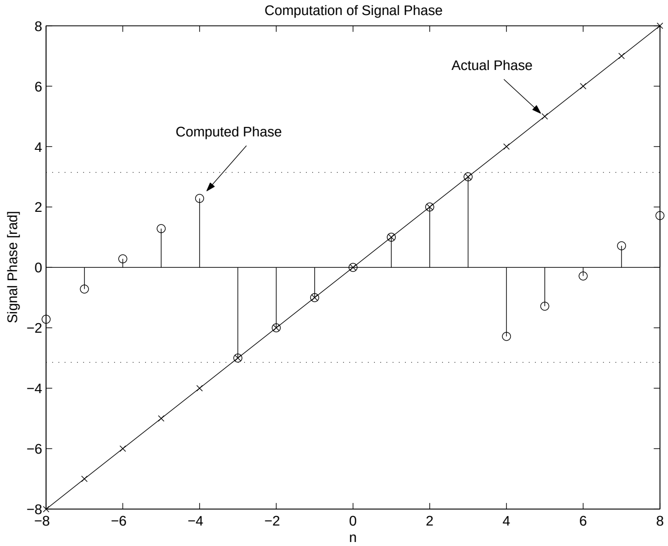

Figure 1: Actual and computed phase for $-8 \leq n \leq 8$ in Equation (5)

phase unwrapping is to search the phase sequentially for jumps in phase greater than π, the assumption being that the phase changes at a rate slower than π radians per sample. These jumps are then corrected by adding a factor of $\pm 2\pi$ to all subsequent terms in the sequence. It is important to note that errors in unwrapping will occur whenever Φ_n changes by more than π radians between samples. Thus, the only phase functions that can be unwrapped successfully are those that obey

$$|\Phi_n - \Phi_{n-1}| < \pi \qquad \text{for all } n \tag{6}$$

The importance of this equation will become evident as we examine certain techniques for phase estimation.

3 Linear Phase

3.1 Problem Statement

A complex sinusoid $s[n; \boldsymbol{\theta}]$ is embedded in complex white Gaussian noise z_n with variance σ^2 (the variance of a complex R.V. is defined in [5], Chapter 15). The signal is given by

$$s[n; \boldsymbol{\theta}] = A \mathrm{e}^{j(2\pi f n + \phi)} \tag{7}$$

where $\boldsymbol{\theta} = [f \;\; \phi]^T$. The parameter A is the signal amplitude, while f, and ϕ are the frequency, and initial phase, respectively. The noise corrupted signal

$$x_n = A \mathrm{e}^{j(2\pi f n + \phi)} + z_n \tag{8}$$

is observed for $n = n_0, n_0 + 1, \ldots, n_0 + N - 1$. We wish to estimate $\boldsymbol{\theta}$, while A is considered a nuisance parameter. It is further known that

$$0 < f < 1$$

$$-\pi < \phi < \pi$$

3.2 Cramer-Rao Lower Bound

The Cramer-Rao Lower Bound (CRLB) for the frequency and initial phase of a complex sinusoid (assuming that $n = 0, 1, \ldots, N - 1$) are given by [3]

$$\sigma^2_{\hat{f}} = \frac{6\sigma^2}{A^2 N(N^2 - 1)(2\pi)^2} \tag{9}$$

$$\sigma^2_{\hat{\phi}} = \frac{\sigma^2(2N - 1)}{A^2 N(N + 1)} \tag{10}$$

The bounds on the variance of the frequency and phase estimators are inversely proportional to SNR (σ^2/A^2). Note also that the CRLB falls off as $1/N^3$ for the frequency estimator while it falls off as as $1/N$ for the phase estimator.

3.3 ML Solution to Estimation of Sinusoidal Parameters

The maximum likelihood (ML) solution to the estimation of sinusoidal parameters is well known (see [3], Chapter 13). The MLE of sinusoidal frequency f is given by

$$\hat{f} = \arg\max_{f} \frac{1}{N} \left| \sum_{n=0}^{N-1} x_n e^{-j2\pi f n} \right|^2 \tag{11}$$

The MLE of initial phase is found using the estimated frequency $\hat{f}$ by solving

$$\hat{\phi} = \tan^{-1} \left[\frac{\mathrm{Im}\left(\sum_{n=0}^{N-1} x_n e^{-j2\pi \hat{f} n} \right)}{\mathrm{Re}\left(\sum_{n=0}^{N-1} x_n e^{-j2\pi \hat{f} n} \right)} \right] \tag{12}$$

Equation (11) should be recognized as the value of f that maximizes the periodogram. Implementation of this approach necessitates the use of numerical methods, which can be implemented in a variety of ways. One typical method is to first perform a coarse grid search over discrete values of f using the FFT to compute the periodogram. Upon finding the general area of the largest peak, a more precise search for the global maxima can be conducted using either a finer grid search, Newton's method, or interpolation. In this paper we use an interpolation procedure involving the peak of the coarse grid search and its two nearest neighbors. Let the three ordered pairs (f_1, a_2), (f_2, a_2), and (f_3, a_3) specify the frequency and amplitude of the point to the left of the peak, the peak itself, and the point to the right of peak, respectively. It is assumed that the periodogram is approximately quadratic in the vicinity of the peak, and can thus be written as

$$a = p_1 f^2 + p_2 f + p_3 \tag{13}$$

The parameters of the equation can be found to be

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} f_1^2 & f_1 & 1 \\ f_2^2 & f_2 & 1 \\ f_3^2 & f_3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{14}$$

The estimated frequency is then approximately the location of the maxima of (13), or

$$\hat{f} = -\frac{p_2}{2p_1} \tag{15}$$

3.4 Alternative Estimation Procedures

The MLE, although very accurate, is quite computationally intensive. Therefore, a variety of sub-optimal but computationally efficient phase estimators have been proposed. We will describe two such estimators here. Tretter proves in [4] that the model for a complex sinusoid in complex WGN noise given in Equation (8) can be approximated at high SNR by a complex sinusoid with real WGN in the phase. That is, the signal model

$$x_n = A \exp\left[j(2\pi f n + \phi)\right] + z_n \quad n = n_0, n_0+1, \ldots, n_0+N-1 \tag{16}$$

where z_n is complex white Gaussian noise, can be approximated at high SNR by

$$x_n \approx A \exp\left[j(2\pi f n + \phi + w_n)\right] \quad n = n_0, n_0+1, \ldots, n_0+N-1 \tag{17}$$

where w_n is real white Gaussian noise. Tretter notes that if given the phase of the expression in in (17), the parameters (f and ϕ) are linear in the data, and are therefore easy to estimate using least squares techniques. Assume we are given access to the instantaneous signal phase

$$\Phi_n = 2\pi f n + \phi + w_n \tag{18}$$

Let

$$\begin{aligned}
\boldsymbol{\Phi} &= [\Phi_{n_0} \ \Phi_{n_0+1} \ \cdots \ \Phi_{n_0+N-1}]^T \\
\boldsymbol{h}_1 &= [2\pi n_0 \ 2\pi(n_0+1) \ \cdots \ 2\pi(n_0+N-1)]^T \\
\boldsymbol{h}_2 &= [1 \ 1 \ \cdots \ 1]^T \\
\boldsymbol{H} &= [\boldsymbol{h}_1 \ \boldsymbol{h}_2]^T
\end{aligned}$$

The least squares solution is given by (see [5], Chapter 8)

$$\begin{bmatrix} \hat{f} \\ \hat{\phi} \end{bmatrix} = (\boldsymbol{H}^T\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{\Phi} \tag{19}$$

In order to implement this estimator, the phase Φ_n must be computed. Tretter suggests finding the angle of the signal using an inverse tangent and then applying standard unwrapping techniques to find the phase. Clearly, unwrapping problems will cause the approach to experience errors at high frequency and/or low SNR (i.e. when the restriction in Equation (6) is violated). It should be noted, however, that the computational requirements are less than a comparable MLE implementation.

In [6], Kay uses the approximation in Equation (17) to derive an estimator of signal frequency that is more computationally efficient than Tretter's approach. Instead of performing least squares on the signal phase, Kay does an MLE on phase differences. Let the phase difference $\Delta\Phi_n$ be defined as

$$\Delta\Phi_n = \Phi_{n+1} - \Phi_n \qquad n = n_0,\ n_0+1,\ \ldots,\ n_0+N-2 \tag{20}$$

The phase can be differenced before computing the argument of the signal to avoid the computational burden of phase unwrapping using

$$\Delta\Phi_n = \angle {x^*}_n x_{n+1} \qquad n = n_0,\ n_0+1,\ \ldots,\ n_0+N-2 \tag{21}$$

Plugging (17) into (21), we find

$$\begin{aligned} \Delta\Phi_n &= \angle {x^*}_n x_{n+1} \\ &= 2\pi f + \Delta w_n \qquad n = n_0,\ n_0+1,\ \ldots,\ n_0+N-2 \end{aligned} \tag{22}$$

where $\Delta w_n = w_{n+1} - w_n$. The problem is now one of estimating a DC level in *colored* Gaussian noise (assuming, again, that the approximation in (17) is valid, i.e. for a high enough SNR). The MLE of f, after some algebra, is found to be a weighted average of phase differences.

$$\hat{f} = \frac{1}{2\pi} \sum_{n=n_0}^{n_0+N-2} h_n \left(\angle {x^*}_n x_{n+1}\right) \tag{23}$$

where

$$h_n = \frac{1.5N}{N^2-1}\left(1 - \left[\frac{n-(N/2-1)}{N/2}\right]\right) \tag{24}$$

The performance of the smoothed phase difference estimator in (23) is identical to the least squares estimator in (19), but since the computational complexity is less, it is to be preferred.

3.5 Comparison of Estimator Performance

Computer simulations were performed to compare the performance of the estimators described in the previous sections. The least squares estimator given in Equation (19) is referred to in the figures as "Tretter's Estimator", while weighted phase difference estimator in (23) referred to as "Kay's estimator". A data record length of $N = 24$ samples was used, and the mean square error was determined using $M = 1000$ trials at each SNR. SNR values from -10 to 30 dB, spaced every 0.5

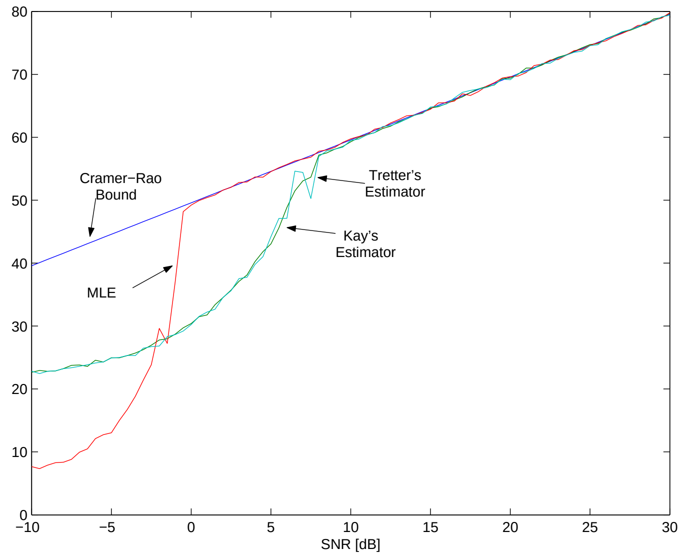

Figure 2: Performance of frequency estimators for $f = 0.05$

dB, were used in the experiment. The mean square errors of Tretter's and Kay's estimators were compared to the mean square error of the MLE, given in Equation (11), and to the Cramer-Rao lower bound (the minimum variance of an unbiased estimator). Two different frequencies were simulated: Figure 2 shows the estimator mean square errors for $f = 0.05$, while Figure 3 shows the estimator mean square errors for $f = 0.35$.

We can see immediately that the performance of Tretter's and Kay's estimators are indeed identical as we expect. Also note that the MLE has an SNR threshold of about -1 dB for both frequencies, while the variance of the other estimators depart from the CRLB at higher SNRs—at about 8 dB when $f = 0.05$, and at about 13 dB when $f = 0.35$.

4 Quadratic Phase

4.1 Problem Statement

A chirp signal $s[n;\boldsymbol{\theta}]$ is embedded in complex white Gaussian noise z_n with variance σ^2. The signal is given by

$$s[n;\boldsymbol{\theta}] = Ae^{j(2\pi(\frac{\alpha}{2}n^2+fn)+\phi)} \tag{25}$$

where $\boldsymbol{\theta} = [\alpha \;\; f \;\; \phi]^T$. The parameter A is the signal amplitude, and α, f, and ϕ are the frequency rate, initial frequency, and initial phase, respectively. The noise corrupted signal

$$x_n = Ae^{j(2\pi(\frac{\alpha}{2}n^2+fn)+\phi)} + z_n \tag{26}$$

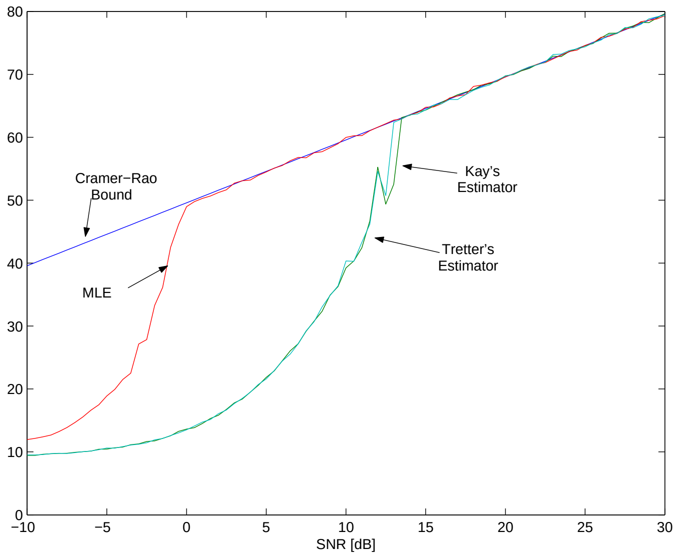

Figure 3: Performance of frequency estimators for $f = 0.35$

is observed for $n = n_0, n_0 + 1, \ldots, n_0 + N - 1$. We wish to estimate $\boldsymbol{\theta}$, while A is considered a nuisance parameter. It is further known that

$$-0.5 < \alpha < 0.5$$

$$0 < f < 1$$

$$-\pi < \phi < \pi$$

4.2 Cramer-Rao Lower Bound

The Cramer-Rao Lower Bound (CRLB) is given by [7]

$$\sigma^2_{\hat{\alpha}} = \frac{\sigma^2}{A^2} \frac{1}{2\pi^2} \frac{QN - P^2}{SQN + 2RPQ - Q^3 - P^2 S - R^2 N} \tag{27}$$

$$\sigma^2_{\hat{f}} = \frac{\sigma^2}{A^2} \frac{1}{8\pi^2} \frac{SN - Q^2}{SQN + 2RPQ - Q^3 - P^2 S - R^2 N} \tag{28}$$

$$\sigma^2_{\hat{\phi}} = \frac{\sigma^2}{2A^2} \frac{SQ - R^2}{SQN + 2RPQ - Q^3 - P^2 S - R^2 N} \tag{29}$$

where

$$P = \sum_{n=n_0}^{n_0+N-1} n, \qquad Q = \sum_{n=n_0}^{n_0+N-1} n^2$$

$$R = \sum_{n=n_0}^{n_0+N-1} n^3, \qquad S = \sum_{n=n_0}^{n_0+N-1} n^4$$

Note that for each parameter, the CRLB is inversely proportional to SNR (σ^2/A^2). It can be seen from (27)–(29) that the CRLB falls off as $1/N^5$, $1/N^3$, and $1/N$ for estimators of α, f, and ϕ respectively.

4.3 ML Solution to Estimation of Chirp Parameters

Here, we derive the MLE of the chirp parameters. The derivation follows the same form as the derivation of complex sinusoid parameters given in [3], Chapter 13.

The probability of observing a data sample x_n is given by

$$p(x_n;\boldsymbol{\theta}) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2\sigma^2}\left|x_n - s[n;\boldsymbol{\theta}]\right|^2\right\} \tag{30}$$

where $s[n;\boldsymbol{\theta}]$ is defined in Equation (25) and x_n is defined in Equation (26). Since the data samples are independent the probability of observing the sequence $\boldsymbol{x} = [x[n_0], x[n_0+1], \ldots, x[n_0+N-1]]^T$ is

$$p(\boldsymbol{x};\boldsymbol{\theta}) = \frac{1}{(2\pi\sigma^2)^{N/2}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{n=n_0}^{n_0+N-1}\left|x_n - s[n;\boldsymbol{\theta}]\right|^2\right\} \tag{31}$$

To maximize the likelihood, we must minimize

$$J(\boldsymbol{\theta}) = \sum_{n=n_0}^{n_0+N-1}\left|x_n - A\mathrm{e}^{j(2\pi(\frac{\alpha}{2}n^2+fn)+\phi)}\right|^2 \tag{32}$$

or, in vector notation

$$J(\boldsymbol{\theta}) = (\boldsymbol{x}-\boldsymbol{s})^H(\boldsymbol{x}-\boldsymbol{s}) \tag{33}$$

Let the complex amplitude of the signal be $A_c = A\,e^{j\phi}$ and let

$$\boldsymbol{e} = \left[e^{j(\pi\alpha n_0^2+2\pi f n_0)} \quad e^{j(\pi\alpha(n_0+1)^2+2\pi f(n_0+1))} \quad \cdots \quad e^{j(\pi\alpha(n_0+N-1)^2+2\pi f(n_0+N-1))}\right]^T$$

Equation (33) can then be re-written as

$$J(\boldsymbol{\theta}) = (\boldsymbol{x}-A_c\boldsymbol{e})^H(\boldsymbol{x}-A_c\boldsymbol{e}) \tag{34}$$

It can be shown that the MLE for the phase, frequency, and frequency rate of a complex chirp is quite similar to the MLE for the frequency and phase of a complex sinusoid. The first step is to estimate the frequency f and frequency rate α by maximizing

$$\frac{1}{N}\left|\boldsymbol{e}^H\boldsymbol{x}\right|^2 = \frac{1}{N}\left|\sum_{n=n_0}^{n_0+N-1} x_n\, e^{-j(\pi\alpha n^2+2\pi f n)}\right|^2 \tag{35}$$

Let f_1 and α_1 the values of f and α that maximize the expression above. The MLE of the complex amplitude is then

$$A_{c1} = \frac{\boldsymbol{e}_1^H\boldsymbol{x}}{\boldsymbol{e}_1^H\boldsymbol{e}_1} \tag{36}$$

where $\boldsymbol{e}_1$ is $\boldsymbol{e}$ evaluated at $f = f_1$ and $\alpha = \alpha_1$. To reiterate, the MLE of α and f is given by

$$\arg\max_{\alpha,f} \left\{ \frac{1}{N} \left| \sum_{n=n_0}^{n_0+N-1} x_n \, e^{-j(\pi\alpha n^2 + 2\pi f n)} \right|^2 \right\} \tag{37}$$

This result can also be extended to higher orders of the phase polynomial. In this context, the expression in (35) appears to be what could be described as "polynomial Fourier transform". See reference [1], page 550, for details.

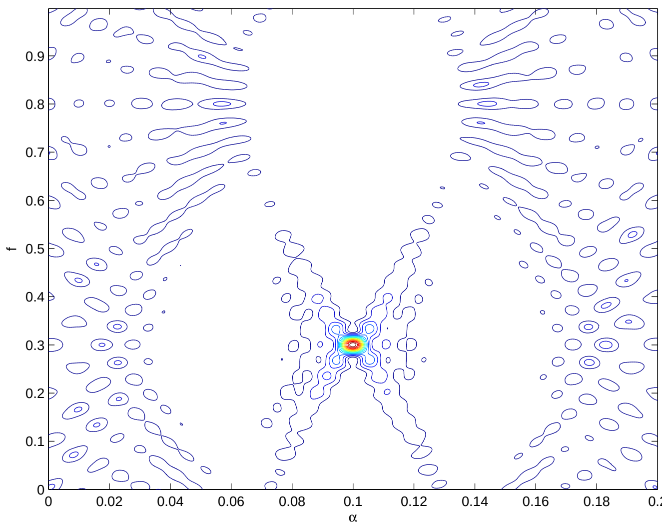

Figure 4: $\alpha = 0.1$, $f = 0.3$, SNR = 20dB

To get a feel for the function to be maximized we have plotted Equation (35) for realizations of x_n at noise levels of 20 and 0 dB in Figures 4 and 5, respectively. In each case, the parameters of chirp signal are: $n_0 = -15$, $N = 31$, $\alpha = 0.1$, $f = 0.3$, and $\phi = 1$. It is encouraging to note that the peak of the function lies at or near the true values of α and f, even at 0 dB SNR. This is an indication of the robustness of the MLE solution. It should also be noted that in Figure 5 there are many strong local maxima in the vicinity of the global maxima—this could (perhaps at a lower SNR) lead to outliers in estimator of Equation (37).

4.4 An Alternative Approach

As was the case for the estimation of sinusoidal parameters, the MLE of chirp parameters is quite computationally intensive, so a variety of suboptimal methods have been proposed. Here we will describe an estimator, proposed by Djuric & Kay, that is an extension of the work in [4] and [6]. Djuric shows that a chirp in white Gaussian noise, as defined in Equation (26), can be approximated by

$$x_n \approx A \exp\left\{ j \left(2\pi \left(\frac{\alpha}{2} n^2 + f n \right) + \phi + w_n \right) \right\} \quad n = n_0, n_0+1, \ldots, n_0+N-1 \tag{38}$$

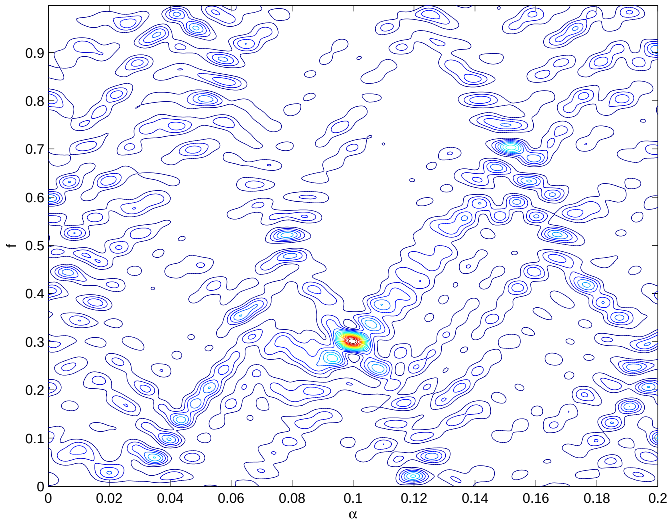

Figure 5: $\alpha = 0.1,\ f = 0.3$, SNR = 0dB

for high SNR. This is similar to the result for a complex sinusoid given in (17). Djuric notes that while the phase of a complex sinusoid is not prone to phase unwrapping errors (unless the frequency f is near $-\frac{1}{2}$ or $\frac{1}{2}$), the phase of a chirp (38) is particularly vulnerable to unwrapping errors since it can change very rapidly for a wide range of frequency rate (α) values. Thus, a scheme for phase unwrapping is needed. A trick can used to pass the yet-to-be-determined phase of the chirp signal through an invertible linear operation before performing any estimation of the phase parameters. This operation greatly reduces problems with unwrapping.

4.4.1 Phase Unwrapping of a Chirp Signal

The goal of the phase unwrapping scheme proposed in [7] is to map the original sequence (38) into a new sequence with a phase that always lies in the region $(-\pi, \pi)$. Then the inverse tangent can then be applied without concern for unwrapping problems. Lastly the transformation is inverted to produce the original phase sequence. The true phase sequence, assuming the approximation in (38), is

$$\Phi_n = \pi\alpha n^2 + 2\pi f n + \phi + w_n \tag{39}$$

We may first apply the backward finite difference

$$\Delta\Phi_n = \Phi_n - \Phi_{n-1} \qquad n = n_0 + 1,\ \ldots,\ n_0 + N - 1 \tag{40}$$

Applying the operation two times we get

$$\begin{aligned} \Delta^2\Phi_n &= \Delta\Phi_n - \Delta\Phi[n-1] \\ &= (\Phi_n - \Phi_{n-1}) - (\Phi_{n-1} - \Phi_{n-2}) \\ &= \Phi_n - 2\Phi_{n-1} + \Phi_{n-2} \qquad n = n_0 + 2,\ \ldots,\ n_0 + N - 1 \end{aligned} \tag{41}$$

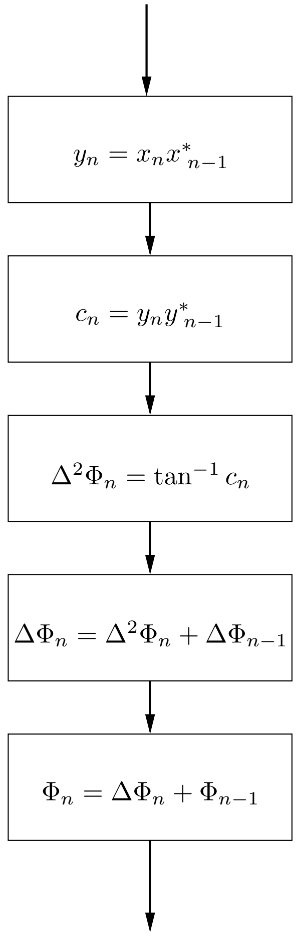

Figure 6: Scheme for phase unwrapping (from [7])

Substituting the value of Φ_n from (39), we have

$$\Delta^2\Phi_n = 2\pi\alpha + \Delta^2 w_n \qquad n = n_0 + 2,\ \ldots,\ n_0 + N - 1 \tag{42}$$

This operation can be implemented before determining the phase using the following steps. The first step is similar to the operation in defined in Equation (21) and is given by

$$y_n = x_n x^*_{n-1} \qquad n = n_0 + 1,\ \ldots,\ n_0 + N - 1 \tag{43}$$

This new sequence y_n is put through the same operation again, producing the sequence

$$c_n = y_n y^*_{n-1} \qquad n = n_0 + 2,\ \ldots,\ n_0 + N - 1 \tag{44}$$

It is a simple matter to show that c_n has the double differenced phase described in (42). The inverse tangent is therefore applied to c_n producing $\Delta^2\Phi_n$. The original phase sequence can then be recovered through double integration.

$$\Delta\Phi_n = \Delta\Phi_{n_0+1} + \sum_{n_0+2}^{n} \Delta^2\Phi_n \qquad n_0+2 \leq n \leq n_0+N-1 \tag{45}$$

$$\Phi_n = \Phi_{n_0} + \sum_{n_0+1}^{n} \Delta\Phi_n \qquad n_0+1 \leq n \leq n_0+N-1 \tag{46}$$

$\Delta\Phi_{n_0+1}$ and Φ_{n_0} are initial conditions of Equations (45) and (46), respectively, and are defined as

$$\Delta\Phi_{n_0+1} = \angle y_{n_0}$$

$$\Phi_{n_0} = \angle x_{n_0}$$

The sequence of steps used for computation of the phase are illustrated in Figure 6. The last two steps are written in their difference equation equivalent form. There is one further point of note: Even a noiseless phase sequence will be in error with this improved unwrapping procedure if the "actual" initial phase or initial phase difference is outside the interval $(-\pi, \pi)$, i.e. if $|\Phi_{n_0}| \geq \pi$ or if $|\Delta\Phi_{n_0+1}| \geq \pi$. Therefore, to avoid such errors we will assume for our experiments that $n_0 = 0$ and choose α and f such that $|\pi\alpha + 2\pi f| < \pi$.

4.4.2 Least Squares Solution

Given the phase sequence Φ_n determined by the special unwrapping procedure described earlier, it is a simple matter to estimate the unknown parameters. Note that the parameters we wish to estimate enter the phase in a linear manner.

$$\Phi_n = \pi\alpha n^2 + 2\pi f n + \phi + w_n \tag{47}$$

The least squares solution to this problem is well know and is given by

$$\begin{bmatrix} \hat{\alpha} \\ \hat{f} \\ \hat{\phi} \end{bmatrix} = (\boldsymbol{H}^T\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{\Phi} \tag{48}$$

where

$$\begin{aligned}
\boldsymbol{\Phi} &= [\Phi_{n_0}\ \Phi_{n_0+1}\ \cdots\ \Phi_{n_0+N-1}]^T \\
\boldsymbol{h}_1 &= [\pi n_0^2\ \pi(n_0+1)^2\ \cdots\ \pi(n_0+N-1)^2]^T \\
\boldsymbol{h}_2 &= [2\pi n_0\ 2\pi(n_0+1)\ \cdots\ 2\pi(n_0+N-1)]^T \\
\boldsymbol{h}_3 &= [1\ 1\ \cdots\ 1]^T \\
\boldsymbol{H} &= [\boldsymbol{h}_1\ \boldsymbol{h}_2\ \boldsymbol{h}_3]^T
\end{aligned}$$

5 Experimental Results

Experiments were conducted using the linear least squares estimator described in Equation (48). An initial sample instance $n_0 = 0$ and a data record length of $N = 31$ samples were used. Chirp parameters were chosen such that the instantaneous frequency of the signal would sweep from 0.1 to 0.4 Hz, a realistic restriction on the form of the chirp. The parameters were: $\alpha = 0.01$, $f = 0.09$, and $\phi = 1$. The mean square errors (MSEs) for each parameter were determined using $M = 1000$ trials for SNRs ranging from -10 to 30 dB in increments of 0.5 dB. The MSEs were then plotted in Figure 7 along side their respective CRLBs defined in equations (27)-(29). One point illustrated quite well by Figure 7 is that, loosely speaking, it

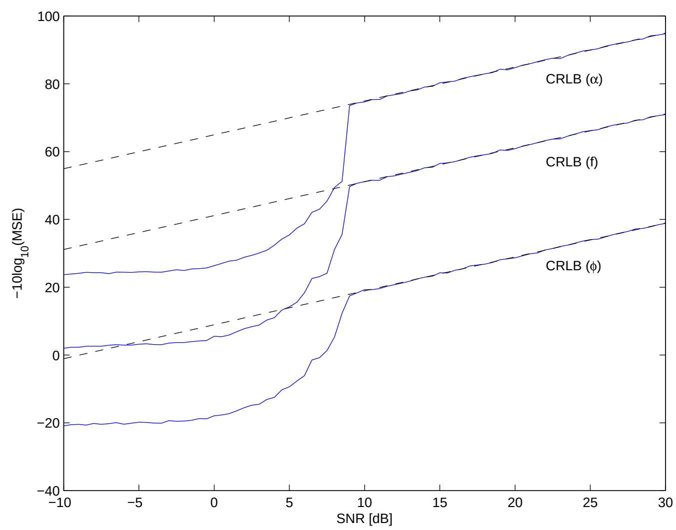

Figure 7: Mean square error vs. CRLB for estimators of the chirp parameters (α, f, and ϕ)

is easier to estimate the frequency rate, α, than the initial frequency, f, which is in turn easier to estimate than the initial phase, ϕ. This is clear from the Cramer-Rao lower bounds alone. (Note that in Figure 7 they are plotted upside-down, i.e. the plotted bound is $-10\log_{10}(\text{CRLB})$.) The the variance of the least squares estimator departs from the CRLB at about 9.5 dB SNR. The performance of the estimator breaks down at this point due to errors in the phase unwrapping algorithm, resulting in a least squares fit to an erroneous phase sequence. On the other hand, the ML approach to this problem (see Section 4.3) does not require the determination of the phase sequence to estimate the unknown parameters. Although no MLE was designed for these experiments, it was noted while creating Figures 4 and 5 that there were no significant outliers for SNRs as low as 0 dB. Thus, as long a particular implementation of the MLE is capable of consistently locating the global maxima of the function in (35), its performance will be superior to the least squares approach of Section 4.4. The drawback is that any implementation of the MLE is likely to be much more computationally intensive than the least squares approach.

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