Problem 4.1

(a) The impulse response of the matched filter is
\[ h(t) = s(T-t) \]
The s(t) and h(t) are shown below:

(b) The corresponding output of the matched filter is obtained by convolving h(t) with s(t). The result is shown below:

(c) The peak value of the filter output is equal to \( A^2 T/4 \), occurring at \( t = T \).
Ideal low-pass filter with variable bandwidth. The transfer function of the matched filter for a rectangular pulse of duration \( \tau \) and amplitude \( A \) is given by

\[
H_{\text{opt}}(f) = \text{sinc}(f\tau)\exp(-j\pi f\tau)
\]  

The amplitude response \( |H_{\text{opt}}(f)| \) of the matched filter is plotted in Fig. 1(a). We wish to approximate this amplitude response with an ideal low-pass filter of bandwidth \( B \). The amplitude response of this approximating filter is shown in Fig. 1(b). The requirement is to determine the particular value of bandwidth \( B \) that will provide the best approximation to the matched filter.

We recall that the maximum value of the output signal, produced by an ideal low-pass filter in response to the rectangular pulse occurs at \( t = T/2 \) for \( BT \leq 1 \). This maximum value, expressed in terms of the sinc integral, is equal to \( (2A/\pi)\text{Si}(\pi BT) \). The average noise power at the output of the ideal low-pass filter is equal to \( BN_0 \). The maximum output signal-to-noise ratio of the ideal low-pass filter is therefore

\[
(SNR)_o = \frac{(2A/\pi)^2\text{Si}^2(\pi BT)}{BN_0}
\]  

Thus, using Eqs. (1) and (2), and assuming that \( AT = 1 \), we get

\[
\frac{(SNR)'_o}{(SNR)_o} = \frac{2}{\pi^2 BT} \text{Si}^2(\pi BT)
\]

This ratio is plotted in Fig. 2 as a function of the time-bandwidth product \( BT \). The peak value on this curve occurs for \( BT = 0.685 \), for which we find that the maximum signal-to-noise ratio of the ideal low-pass filter is 0.84 dB below that of the true matched filter. Therefore, the "best" value for the bandwidth of the ideal low-pass filter characteristic of Fig. 1(b) is \( B = 0.685/T \).
Problem 4.6

The average probability of error is

\[ P_e = p_1 \int_{-\infty}^{\lambda} f_Y(y \mid 1) \, dx + p_0 \int_{\lambda}^{\infty} f_Y(y \mid 0) \, dx \]  

(1)

An optimum choice of \( \lambda \) corresponds to minimum \( P_e \). Differentiating Eq. (1) with respect to \( \lambda \), we get:

\[ \frac{\partial P_e}{\partial \lambda} = p_1 f_Y(\lambda \mid 1) - p_0 f_Y(\lambda \mid 0) \]

Setting \( \frac{\partial P_e}{\partial \lambda} = 0 \), we get the following condition for the optimum value of \( \lambda \):

\[ \frac{f_Y(\lambda_{\text{opt}} \mid 1)}{f_Y(\lambda_{\text{opt}} \mid 0)} = \frac{p_0}{p_1} \]

which is the desired result.
Problem 4.8

(a) The average probability of error is

\[ P_e = \frac{1}{2} \text{erfc}\left(\frac{E_b}{\sqrt{N_0}}\right) \]

where \( E_b = A^2 T_b \). We may rewrite this formula as

\[ P_e = \frac{1}{2} \text{erfc}\left(\frac{A}{\sigma}\right) \quad (1) \]

where \( \Lambda \) is the pulse amplitude at \( \sigma = \sqrt{N_0 T_b} \). We may view \( \sigma^2 \) as playing the role of noise variance at the decision device input. Let

\[ u = \frac{E_b}{\sqrt{N_0}} = \frac{A}{\sigma} \]

We are given that

\[ \sigma^2 = 0.2 \text{ volts}^2, \quad \sigma = 0.1 \text{ volt} \]

\[ P_o = 10^{-8} \]

Since \( P_o \) is quite small, we may approximate it as follows:

\[ \text{erfc}(u) = \frac{\exp(-u^2)}{\sqrt{\pi} u} \]
We may thus rewrite Eq. (1) as (with $P_0 = 10^{-6}$)

$$\frac{\exp(-u^2/2\sqrt{\pi u})}{u} = 10^{-8}$$

Solving this equation for $u$, we get

$$u = 3.97$$

The corresponding value of the pulse amplitude is

$$A = \sigma u = 0.1 \times 3.97$$
$$= 0.397 \text{volts}$$

(b) Let $\sigma_1^2$ denote the combined variance due to noise and interference; that is

$$\sigma_T^2 = \sigma^2 + \sigma_1^2$$

where $\sigma^2$ is due to noise and $\sigma_1^2$ is due to the interference. The new value of the average probability of error is $10^{-6}$, Hence

$$10^{-6} = \frac{1}{2} \text{erfc} \left( \frac{A}{\sigma_T} \right)$$

$$= \frac{1}{2} \text{erfc}(u_T)$$

where

$$u_T = \frac{A}{\sigma_T}$$

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Equation (2) may be approximated as (with $P_e = 10^{-6}$)

$$\frac{\exp\left(-\frac{u_T^2}{2\sqrt{\pi} u_T}\right)}{2\sqrt{\pi} u_T} = 10^{-6}$$

Solving for $u_T$, we get

$$u_T = 3.37$$

The corresponding value of $\sigma_T^2$ is

$$\sigma_T^2 = \left(\frac{\Lambda}{u_T}\right)^2 \left(\frac{0.397}{3.37}\right)^2 = 0.0138 \text{ volts}^2$$

The variance of the interference is therefore

$$\sigma_i^2 = \sigma_T^2 - \sigma^2$$

$$= 0.0138 - 0.01$$

$$= 0.0038 \text{ volts}^2$$
Problem 4.13

Since \( P(f) \) is an even real function, its inverse Fourier transform equals

\[
p(t) = 2 \int_0^\infty P(f) \cos(2\pi ft) \, df
\]

(1)

The \( P(f) \) is itself defined by Eq. (7.60) which is reproduced here in the form

\[
P(f) = \begin{cases} 
\frac{1}{2W}, & 0 < |f| < f_L \\
\frac{1}{4W} \left( 1 + \cos \left( \frac{\pi |f| - f_L}{2W - 2f_L} \right) \right) & f_L < f < 2W-f_L \\
0, & |f| > 2W-f_L
\end{cases}
\]

(2)

Hence, using Eq. (2) in (1):

\[
p(t) = \frac{1}{W} \int_0^{f_L} \cos(2\pi ft) \, df + \frac{1}{2B} \int_0^{2W-f_L} \left[ 1 + \cos \left( \frac{\pi (f-f_L)}{2W} \right) \right] \cos(2\pi ft) \, df
\]

\[
= \left[ \frac{\sin(2\pi ft)}{2\pi Wt} \right]_0^{f_L} + \left[ \frac{\sin(2\pi ft)}{4\pi Wt} \right]_0^{2W-f_L}
\]

\[
+ \frac{1}{4} \left[ \frac{\sin \left( 2\pi ft + \frac{\pi (f-f_L)}{2W} \right)}{2\pi t + \pi/2W} \right]_0^{2W-f_L}
\]

\[
+ \frac{1}{4} \left[ \frac{\sin \left( 2\pi ft - \frac{\pi (f-f_L)}{2W} \right)}{2\pi t - \pi/2W} \right]_0^{2W-f_L}
\]

\[
= \frac{\sin(2\pi f_L t)}{4\pi Wt} + \frac{\sin(2\pi f_L t)}{4\pi Wt}
\]

\[
- \frac{1}{4W} \left[ \frac{\sin(2\pi f_L t) + \sin(2\pi t/(2W-f_L))}{2\pi t - \pi/2W} \right] + \frac{\sin(2\pi f_L t) + \sin(2\pi t/(2W-f_L))}{2\pi t - \pi/2W}
\]

\[
= \frac{1}{W} \left[ \sin(2\pi f_L t) + \sin(2\pi t(2W-f_L)) \right] \left[ \frac{1}{4\pi t} - \frac{\pi t}{(2\pi t)^2 - (\pi/2W)^2} \right]
\]
\[
\frac{1}{W} \left[ \sin(2\pi W t) \cos(2\pi \alpha W) \right] \left[ \frac{-(\pi/2W\alpha)^2}{4\pi^2 ((2\pi t)^2 - (\pi/2W\alpha)^2)} \right] \\
= \sin(2Wt) \cos(2\pi \alpha W) \left[ \frac{1}{1 - 16 \alpha^2 W^2 t^2} \right]
\]

Problem 4.16

The bandwidth \( B \) of a raised cosine pulse spectrum is \( 2W = f_1 \), where \( W = 1/2T_0 \) and \( f_1 = W(1-\alpha) \). Thus \( B = W(1+\alpha) \). For a data rate of 56 kilobits per second, \( W = 28 \) kHz.

(a) For \( \alpha = 0.25 \),

\[
B = 28 \text{ kHz} \times 1.25 \\
= 35 \text{ kHz}
\]

(b) \( B = 28 \text{ kHz} \times 1.5 \)

\[
= 42 \text{ kHz}
\]

(c) \( B = 49 \text{ kHz} \)

(d) \( B = 56 \text{ kHz} \)

Problem 4.17

The use of eight amplitude levels ensures that 3 bits can be transmitted per pulse. The symbol period can be increased by a factor of 3. All four bandwidths in problem 7.12 will be reduced to \( 1/3 \) of their binary PAM values.

Problem 4.19

The raised cosine pulse bandwidth \( B = 2W - f_1 \), where \( W = 1/2T_0 \). For this channel, \( B = 75 \) kHz. For the given bit duration, \( W = 50 \) kHz. Then,

\[
f_1 = 2W - B \\
= 25 \text{ kHz} \\
\alpha' = 1 - f_1/B \\
= 0.5
\]