\[
x = \sqrt{\frac{\sigma^2}{2}} \ln \left( \frac{1 - e^{-\frac{1}{2} \ln^2 \frac{x}{\sigma \sqrt{2 \pi}}} - \frac{1}{2}}{\sigma \sqrt{2 \pi}}\right), \quad 0 \leq x \leq \frac{1}{2} \\
\text{NEGATIVE VALUES OF X}
\]

\[
\text{POSITIVE VALUES OF X}
\]

Figure 10.30: Computer generation of Laplacian random variable outcomes using inverse probability integral transformation.

---

**ESTIMATING THE PDF**

RECALL \[ P_X(x_0) = \frac{P \left[ x_0 - \Delta x \leq x \leq x_0 + \Delta x \right]}{\Delta x} \]

BUT \[ P \left[ x_0 - \frac{\Delta x}{2} \leq x \leq x_0 + \frac{\Delta x}{2} \right] \approx \frac{\text{NUMBER OF OUTCOMES IN } [x_0 - \frac{\Delta x}{2}, x_0 + \frac{\Delta x}{2}]}{M} \]

\[ \Rightarrow P_X(x_0) = \frac{\text{NUMBER OF OUTCOMES IN } [x_0 - \frac{\Delta x}{2}, x_0 + \frac{\Delta x}{2}]}{M \Delta x} \]
IN PREVIOUS EXAMPLE $\Delta x = 0.5$ AND
BINS ARE $[-4.25, -3.75], [-3.75, -3.25], \ldots ,$
$[3.75, 4.25] \uparrow \uparrow \uparrow \uparrow \uparrow \hat{p}_x(-4.5) \hat{p}_x(-3.5)$
$\hat{p}_x(4.5)$

SEE MATLAB CODE ON PG. 328 AND EXAMPLE 2.1

CHAPTER 11 - EXPECTED VALUES

FOR A DISCRETE R.V. $E[x] = \sum_{i} x_i p(x_i)$
(SEE SECTION 6.3)

EXAMPLE: FAIR DIE TOSS
$S_x = \{1, 2, 3, 4, 5, 6\}$

$E(x) = \frac{6}{2} = \frac{1}{6} (1 + 2 + 3 + 4 + 5 + 6)$
$= \frac{1}{2}$

FOR A CONT. R.V. WE DEFINE $E(x)$ AS

$E(x) = \int_{-\infty}^{\infty} x p(x) dx$

CAN BE THOUGHT OF AS LIMIT OF
\[ \sum_{i=1}^{n} x_i p(x_i) \Delta x \quad \text{as } \Delta x \to 0 \]

\[ \approx p \left[ \frac{x_i - \Delta x/2}{\Delta x} \leq x_i + \Delta x/2 \right] \]

\[ \approx p(x_i) \]

See book for discrete to continuous definition example.

**Example:** \( x \sim U(0, 1) \)

\[ E[x] = \int_0^1 x \cdot 1 \, dx = \frac{x^2}{2} \bigg|_0^1 = \frac{1}{2} \]

**Example:**

\[ E[x] = \int_0^2 x^2 \cdot \frac{x}{2} \, dx \]

\[ = \left. \frac{x^3}{2} \right|_0^2 = \frac{8}{2} = 4 \]

\[ = \frac{4}{2} = 1.83 \]

(a) PDF

(b) Typical outcomes and expected value of 1.33

Figure 11.1: Example of nonuniform PDF and its mean.

**Analogous to center of mass**

\[ CM = \int_0^2 x m(x) \, dx \]

\[ \uparrow \text{Mass density } = \frac{\Delta m}{\Delta x} \]

As \( \Delta x \to 0 \)
TOTAL VOLUME = 1

\[ M(x) = \lim_{\Delta x \to 0} \frac{\Delta M}{\Delta x} \]

\[ \Delta V = \Delta A \times \text{AREA} \]

\[ \Delta A = \frac{1}{2} \Delta x \left( \frac{x - \Delta x}{2} + \frac{x + \Delta x}{2} \right) = \frac{1}{2} x \Delta x \]

\[ \Delta y = \frac{x}{2} \Delta x \]

But \( \Delta = \frac{M}{V} = 1 \Rightarrow \Delta M = \Delta V \)

\[ \frac{\Delta M}{\Delta x} = \Delta y = \frac{1}{2} x \text{ and as } \Delta x \to 0 \]

\[ \frac{\Delta M}{\Delta x} \to \frac{1}{2} x = m(x) \]

\[ \left[ m(x) \right] \]

\[ \int_{0}^{2} x m(x) \, dx = E(x) \]

Says we can balance cheese at \( x = 4/3 \) or \( \int_{0}^{2} (x - CM) \frac{m(x)}{M} \, dx = 0 \)

\[ \int_{-\infty}^{\infty} (x - E[x]) p(x) \, dx = 0 \]
IF \( p_x(\cdot) \) SYMMETRIC ABOUT \( \mu = \alpha \)  \\
\( E[X] = \alpha \). CONVERSE TRUE?

NOT ALL PDFS HAVE EXPECTED VALUES. TRY SIMULATING \( p_x(x) = \begin{cases} 
\frac{1}{2x^{3/2}} & x > 1 \\
0 & x < 1 
\end{cases} \)

AND AVERAGING VALUES. FOR \( E(X) \) TO EXIST REQUIRE \( \int_0^\infty |x| p_x(x) dx < \infty \)

EXPECTED VALUES FOR IMPORTANT PDFS

1) \( X \sim U(a, b) \)  \( E[X] = \frac{1}{2}(a + b) \)

2) \( X \sim Exp(\lambda) \)  \\
\( E[X] = \int_0^\infty X \lambda e^{-\lambda x} dx \)  \\
\[ = \left[ -\lambda e^{-\lambda x} - \frac{1}{\lambda} e^{-\lambda x} \right]_0^\infty = \frac{1}{\lambda} \]

(SEE TABLE OF INTEGRALS, SERIES, 
AND PRODUCTS BY GRADSHTEYN 
AND RYZHIK, ACADEMIC PRESS, 1994)

3) \( X \sim N(\mu, \sigma^2) \)  \\
PDF IS SYMMETRIC ABOUT \( \mu = \alpha \)  \\
\( \Rightarrow E[X] = \mu \) ALSO CALLED MEAN 
OR AVERAGE VALUE
A direct computation yields

\[ E(x) = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi} \sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx \]

\[ = \int_{-\infty}^{\infty} (x-\mu) \frac{1}{\sqrt{2\pi} \sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx \]

\[ + \int_{-\infty}^{\infty} \mu \frac{1}{\sqrt{2\pi} \sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx \]

Let \( u = x-\mu \) in first integral

\[ = \int_{-\infty}^{\infty} u \frac{1}{\sqrt{2\pi} \sigma^2} e^{-\frac{u^2}{2\sigma^2}} \, du = 0 \]

Odd \quad Even

Second integral = \( \mu \) \quad Why?

4) Laplacian

\[ p_X(x) = \frac{1}{\sqrt{2\sigma^2}} e^{-\sqrt{2\sigma^2} |x|} \quad -\infty < x < \infty \]

\[ \Rightarrow E(x) = 0 \quad \text{Why?} \]

See others in book

Expected value of \( y = g(x) \)

Assume \( x \) and \( y \) are cont. r.v.s

By definition \( E(y) = \int_{-\infty}^{\infty} y \, p_Y(y) \, dy \)
REQUIRES US TO FIND $\mathbb{E}(y)$. TO AVOID THIS, USE

$$E(g(x)) = \int_{-\infty}^{\infty} g(x) p_x(x) \, dx$$

VERY USEFUL! SEE APPENDIX IIA FOR PROOF.

(IF $g(x) = x \Rightarrow$ DEFINITION OF $E(x)$).

**Example**: $g(x) = ax + b$

$$E(g(x)) = \int_{-\infty}^{\infty} (ax + b) p_x(x) \, dx$$

$$= a \underbrace{\int_{-\infty}^{\infty} x p_x(x) \, dx}_{E(x)} + b \underbrace{\int_{-\infty}^{\infty} p_x(x) \, dx}_{1} = 1$$

$$= a E(x) + b$$

In general, $E(a_1 g_1(x) + a_2 g_2(x)) = a_1 E(g_1(x)) + a_2 E(g_2(x))$ \Rightarrow **Expectation operation is linear**.

**Example**: $y = x^2$, $x \sim N(0,1)$

$E(y) = \frac{\text{Average Power Across 1 Ohm Resistor}}{x = \text{Voltage}}$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx$$
USE INTEGRATION BY PARTS

\[ \int u \, dv = uv - \int v \, du \]

\[ u = x, \quad dv = x \frac{i}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \]
\[ du = dx, \quad v = -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \]

Also, \[ E(x^2) = 2 \int_0^\infty x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \]

Why?

\[ E(x^2) = 2 \left[ -x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \right]_0^\infty \]
\[ - \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \]
\[ = 2 \left[ 0 + \frac{1}{2} \right] = 1 \]

Note: Limit of \( xe^{-\frac{1}{2}x^2} \) as \( x \to \infty \)

is zero (L'Hospital's Rule)

and \( \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \frac{1}{2} \)

Why?

**VARIANCE AND MOMENTS**

**VARIANCE MEASURES VARIABILITY OF R.V. OUTCOMES**
Figure 10.9: Examples of Gaussian PDF with different $\sigma^2$'s.

Appears as if the wider the PDF the more variability. To measure width define variance

$$\text{VAR}(x) = \int_{-\infty}^{\infty} (x - E(x))^2 p_X(x) \, dx$$

Averaging PDF

= Average squared deviation from mean

Example: $\mathcal{N}(\mu, \sigma^2)$

$$\text{VAR}(x) = \int_{-\infty}^{\infty} (x - E(x))^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx$$
LET \( u = \frac{X - \mu}{\sigma} \)  \( \sigma = \sqrt{\sigma^2} > 0 \)

\[
= \int_{-\infty}^{\infty} \sigma^{-u^2} e^{-\frac{1}{2} u^2} \sigma \, du
\]

\[
= \sigma^{-u^2} \int_{-\infty}^{\infty} u e^{-\frac{1}{2} u^2} \, du = \sigma^{-u^2} E[x^2]
\]

For \( X \sim N(0,1) \)

\[
= \sigma
\]

HENCE \( X \sim N(\mu, \sigma^2) \)

\( \mu \) \( \uparrow \) \( \uparrow \) \( \text{mean variance} \)

\( \sigma = \sqrt{\sigma^2} \) CALLED THE STANDARD DEVIATION

NOTE THAT A \( N(\mu, \sigma^2) \) RV WILL DEVIATE FROM MEAN ABOUT 99.8\% OF TIME

\( \mu - 3\sigma \leq X \leq \mu + 3\sigma \) (USEFUL FOR QUICK ASSESSMENT OF RANGE OF OUTCOMES).

Figure 11.5: Percentage of outcomes of \( N(1,1) \) random variable that are within \( k = 1, 2, \) and 3 standard deviations from the mean. Shaded regions denote area within interval \( \mu - k\sigma \leq x \leq \mu + k\sigma \).
VERIFY THIS:
\[ P \left( |x - 35| \leq 2 \sigma \right) \]
For \( x \sim N(\mu, \sigma^2) \)

**Properties:**

1. \( \text{VAR}(c) = 0 \quad \text{C = CONSTANT} \)
2. \( \text{VAR}(x + c) = \text{VAR}(x) \)
3. \( \text{VAR}(cx) = c^2 \text{VAR}(x) \)

**Note:** \( \text{VAR} \left( g_1(x) + g_2(x) \right) \neq \text{VAR}(g_1(x)) + \text{VAR}(g_2(x)) \)

*Not Linear*

Also, \( \text{VAR}(x) = E \left( x^2 \right) - E^2(x) \)

\( E(x) \) also called **First Moment**
\( E \left( x^r \right) \) = **r-th Moment**

(If \( E \left( x^s \right) < \infty \Rightarrow E \left( x^r \right) < \infty \) for \( r < s \))

(If \( E \left( x^r \right) = \infty \Rightarrow E \left( x^s \right) = \infty \))

If the RV has \( E(x) = \infty \)
\( \Rightarrow E(x^2) = \infty \Rightarrow \text{VAR}(x) \) is not defined.
**Example:** \( X \sim \text{Exp}(\lambda) \)

\[
E(x^n) = \int_0^\infty x^n \lambda e^{-\lambda x} \, dx \quad n = 1, 2, 3, \ldots
\]

**Use integration by parts to find** \( E(x^n) \) **as function of** \( E(x^{n-1}) \) **(standard trick)**

Let \( u = x^n \), \( dv = \lambda e^{-\lambda x} \, dx \)
\[du = nx^{n-1} \, dx \quad v = -e^{-\lambda x}\]

\[
E(x^n) = -x^n e^{-\lambda x} \bigg|_0^\infty - \int_0^\infty -e^{-\lambda x} \, n x^{n-1} \, dx
\]
\[
= 0 - n \int_0^\infty x^{n-1} e^{-\lambda x} \, dx = \frac{n}{\lambda} \int_0^\infty x^{n-1} e^{-\lambda x} \, dx
\]

**Since** \( E(x') = \frac{1}{\lambda} \) **(easy to verify)**

\[
E(x^2) = \frac{2}{\lambda} \frac{1}{\lambda} = \frac{2}{\lambda^2}
\]

\[
E(x^3) = \frac{3}{\lambda^2} \frac{2}{\lambda} = \frac{6}{\lambda^3}
\]

\[
E(x^n) = \frac{n!}{\lambda^n}
\]

**Characteristic functions**

**Useful to find moments, and later will allow us to find PDF of**
Sum of R.V.s

Defined as \( \phi_x(w) = E[e^{jwx}] \)

Recall \( E[g(x)] = \int_{-\infty}^{\infty} g(x) p_x(x) dx \)

\[ \phi_x(w) = E[\cos(wx) + j \sin(wx)] \]
\[ = E[\cos(wx)] + j E[\sin(wx)] \]
\[ = \int_{-\infty}^{\infty} \cos(wx) p_x(x) dx + j \int_{-\infty}^{\infty} \sin(wx) p_x(x) dx \]
\[ = \int_{-\infty}^{\infty} e^{jwx} p_x(x) dx \]

or \( = \int_{-\infty}^{\infty} p_x(x) e^{jwx} dx \)

Continuous Fourier Transform

Recall \( S(t) \leftrightarrow S(w) \)

Now \( p_x(t) \leftrightarrow \phi_x(w) \)

Only difference is use of \( +j \) or \( e^{+jwx} \) instead of \( e^{-jwx} \).

Also, using Fourier Transform Theory

\[ p_x(x) = \int_{-\infty}^{\infty} \phi_x(w) e^{-jwx} \frac{dw}{2\pi} \]
JUST AN INVERSE FOURIER TRANSFORM

TO FIND MOMENTS USING \( \phi_x(w) \):

\[
E(x^n) = \frac{1}{2\pi} \left. \frac{d^n \phi_x(w)}{dw^n} \right|_{w=0}
\]

ALWAYS EASIER TO DIFFERENTIATE THAN INTEGRATE! (ONCE \( \phi_x(w) \) KNOWN)

EXAMPLE: \( x \sim \exp(\lambda) \)

TO FIND \( \phi_x(w) \):

\[
\phi_x(w) = E(e^{jwx}) = \int_{-\infty}^{\infty} p_x(x) e^{jwx} dx = \int_{0}^{\infty} e^{-\lambda e^{-\lambda x} x} dx
\]

\[
= \frac{e^{-j\lambda \omega x}}{-\lambda-j\omega} \bigg|_{0}^{\infty} = -\frac{1}{\lambda-j\omega} \left[ e^{-(\lambda-j\omega) \infty} - 1 \right]
\]

BUT \( \lim_{x \to \infty} e^{-(\lambda-j\omega) x} = 0 \) FOR \( \lambda > 0 \)

WHY?

\[
\phi_x(w) = \frac{1}{\lambda-j\omega} \quad \text{(OR COULD LOOK UP IN TABLES)}
\]

TO FIND \( E(x^n) \) USE FORMULA.
\[
\frac{d \phi_x(w)}{dw} = \frac{d}{dw} \log(1 - g w) = \frac{1}{1 - g w} \cdot \frac{d}{dw} (1 - g w)^{-1} = \frac{1}{1 - g w} \cdot (-g) \cdot (1 - g w)^{-2} = -g (1 - g w)^{-3} \\
\frac{d^2 \phi_x(w)}{dw^2} = \frac{d}{dw} \left( \frac{-g}{1 - g w} \cdot (1 - g w)^{-3} \right) = \frac{g^2}{(1 - g w)^2} \cdot (1 - g w)^{-4} \\
\frac{d^n \phi_x(w)}{dw^n} = \frac{d}{dw} \left( \frac{g^n}{1 - g w} \cdot (1 - g w)^{-n+1} \right) = \frac{g^n}{1 - g w} \cdot (1 - g w)^{-n+2} \\
\text{AT } w = 0 = \frac{g^n}{1 - g w} \cdot (1 - g w)^{-n+1} \cdot (1 - g w)^{-n+2} = \frac{g^n}{1^n} \\
E(x^n) = \frac{1}{g^n} \cdot \frac{d^n \phi_x(w)}{dw^n} \bigg|_{w=0} = \frac{n!}{1^n} \\
\]

**Chebyshev Inequality**

The variance can also be used to bound a probability. Consider finding

\[ P \left( |X - E[X]| > \delta \right) \]

What can be said if we can't integrate \( E(x) - \delta \) or if we don't know \( \phi_x(x) \)?
Assume we know $E(x)$ and $\text{VAR}(x)$ (We will see how to estimate these next!). Then, Chebyshev's Inequality provides a bound $B$ so that

$$P\left(\mid x - E(x)\mid > \delta \right) \leq B$$

Probability of $x$ deviating from mean by more than $\delta$ is less than or equal to $B$.

$$B = \frac{\text{VAR}(x)}{\delta^2}$$

**Example:** $X \sim N(0, 1)$

$E(x) = 0$, $\sigma = 1$, $\text{VAR}(x) = 1$

$$P(\mid x - 0 \mid > 3) \leq \frac{1}{3^2} = \frac{1}{9} = 0.11$$

Actually, $P(\mid x \mid > 3) = 2 P(\mid x \mid > 3) = 2 \phi(3) = 0.0027$

Bound holds but not very "tight" for this example.

**Example:** For a Laplacian PDF with $\sigma^2 = 1 = \text{VAR}(x)$

$$P(\mid x \mid > 3) \leq \frac{1}{3^2} = 0.11$$
SAME BOUND FOR ALL PDFs WITH $\text{VAR}(x) = 1$ (DON'T NEED TO KNOW PDF)

$P(|x| > 3) = 0.0027$ FOR GAUSSIAN

$P(|x| > 3) = 0.0144$ FOR LAPLACIAN

Figure 11.8: Probabilities $P(|X| > \gamma)$ for Gaussian and Laplacian random variables with zero mean and unity variance compared to Chebyshev inequality.

MOST USEFUL FOR THEORETICAL WORK - CAN PROVE THAT AS $\text{VAR}(x) \to 0$,

$P(|x - E(x)| > \gamma) \to 0$ FOR ANY $\gamma > 0$.

PROOF:

$$\text{VAR}(x) = \int_{-\infty}^{\infty} (x - E(x))^2 p(x) dx$$

$$= \int_{\{x: |x - E(x)| > \delta\}} (x - E(x))^2 p(x) dx + \int_{\{x: |x - E(x)| \leq \delta\}} (x - E(x))^2 p(x) dx$$

$$\geq \int_{\{x: |x - E(x)| > \delta\}} (x - E(x))^2 p(x) dx$$

$$\geq \int_{\{x: |x - E(x)| > \delta\}} \delta^2 p(x) dx$$

$$= \delta^2 P(|x - E(x)| > \delta)$$
ESTIMATING MEAN AND VARIANCE

ASSUME X IS A DISCRETE RANDOM VARIABLE

\[ E(x) = \sum_{k=1}^{5} k \cdot p_x(k) \]

**BUT** \[ p_x(k) = P(X = k) \approx \frac{N_k}{M} \]

\[ E(x) \approx \frac{5}{2} \sum_{k=1}^{5} \frac{N_k}{M} = \frac{5}{2} \frac{K N_k}{M} \]

**BUT IF** \[ \{1, 1, 5, 3, 2, 3, 4, 1\} \]

\[ N_1 = 3, \quad N_2 = 1, \quad N_3 = 2, \quad N_4 = 1, \quad N_5 = 1 \]

\[ \frac{5}{2} \sum_{k=1}^{5} k N_k = 1(3) + 2(1) + 3(2) + 4(1) + 5(1) = 20 = \frac{5}{2} \sum_{i=1}^{5} X_i \quad (M = 8) \]
\[ E(x) \approx \frac{\sum x_i}{8} \]

or in general
\[ E(x) = \frac{1}{M} \sum_{i=1}^{M} x_i \]

is the sample mean, and
\[ \text{VAR}(x) = E(x^2) - E(x)^2 \]

so that
\[ \text{VAR}(x) = E(x^2) - \left( \frac{1}{M} \sum_{i=1}^{M} x_i \right)^2 \]

Figure 6.7: Estimated mean and variance for computer data shown in Figure 6.6.

See also Example 2.3.

For a cont. r.v.
\[ E(x) = \int_{-\infty}^{\infty} x f(x) \, dx \]

\[ \approx \sum_{k} x_k P(x_k - \Delta x/2 \leq x \leq x_k + \Delta x/2) \]

\[ \Delta x \approx 2 \times \Delta y \times x_k \]
\[
\sum_{k=1}^{N_k} \frac{x_k}{M} \\
N_k = \text{NUMBER OF } x \text{'S FALLING IN } x_k \text{ INTERVAL}
\]

\[
\frac{1}{M} \sum_{i=1}^{M} x_i^2 \quad (\text{AS BEFORE})
\]

We will justify this more rigorously latter (Law of Large Numbers)

**Chapter 12 - Multiple Cont. Rvs**

We will consider two RVs, \( x \) and \( y \). They are said to be jointly distributed if the original experimental sample space \( S \) maps into two numbers \( x(s) = x \) and \( y(s) = y \) \( s \in S \)

![Diagram](image)

Figure 12.1: Mapping of the outcome of a thrown dart to the plane (example of jointly continuous random variables).

In general, outcomes are now pairs of numbers \((x, y)\) and hence \( S_{x,y} = \{(x, y): -\infty < x < \infty, -\infty < y < \infty\} \)

(For discrete RVs height and weight of individual are jointly distributed)