NON-GAUSSIAN NOISE

(CHAPERT  10)

GAUSSIAN PDF MODEL JUSTIFIED BY CENTRAL LIMIT THEOREM

EXAMPLE: RESISTOR NOISE, ELECTRONS ALL MOVING AT ABOUT SAME VELOCITY \( \Rightarrow \) CURRENT OR CHARGE PER UNIT TIME IS SOME AVERAGE WITH SMALL DEVIATIONS

WHAT HAPPENS IF WE HAVE SOME "HOT RODDING" ELECTRONS?

NOISE IS NON-GAUSSIAN.
Other examples - high level

But infrequent events such as electromagnetic noise spikes due to thunderstorms or acoustic spikes due to iceberg breakup.

Consider now deterministic signals in non-Gaussian noise.

Noise characteristics.

Simplest model is FID.

Example: Laplacian

\[ p(w_{1:n}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(w_{1:n})^2}{2}} \]

All \( w_{1:n} \) have same PDF and are independent.

\[ p(w) \]

Gaussian \( e^{-\alpha w^2} \)

\[ e^{-\beta w^2} \]
*Laplacian PDF tails are heavier \( \Rightarrow \) more high level events*

\[ a) \]

**Figure 10.1 - Gaussian versus Typical Non-Gaussian PDF \( \sigma^2 = 1 \)**

*Note: Both PDFs have same variance \( \sigma^2 \) - otherwise comparing "apples and oranges"*
\textit{Figure 10.1}
Figure 10.2 - Realization of Gaussian and Non-Gaussian Noise Processes ($\sigma^2 = 1$)

High level events called **spikes or outliers** - can increase PFA for Gaussian designed detectors (need limiters or clippers)
GENERALIZED GAUSSIAN PDFS

ENCOMPASSES LAPLACIAN, GAUSSIAN, UNIFORM

\[ p(w) = \frac{c_1(\beta)}{\sqrt{\sigma^2}} e^{-\frac{c_2(\beta)}{\sqrt{\sigma^2}} \left| \frac{w}{\sigma} \right|^{1+\beta}} \]

\[ -\infty < w < \infty \]

WHERE

\[ c_1(\beta) = \Gamma\left(\frac{1}{2}\right) \left(\frac{\Gamma\left(\frac{1}{2}(1+\beta)\right)}{(1+\beta) \Gamma\left(\frac{1}{2}(1+\beta)\right)}\right)^{-1}\]

\[ c_2(\beta) = \left[ \frac{\Gamma\left(\frac{3}{2}(1+\beta)\right)}{\Gamma\left(\frac{1}{2}(1+\beta)\right)} \right]^{1+\beta} \]

\[ -1 < \beta < 1 \] AND DETERMINES DEGREE OF NON-GAUSSIANITY

EXAMPLE:

\[ \beta = 0 \Rightarrow \text{GAUSSIAN} \]

\[ \beta = 1 \Rightarrow \text{LAPLACIAN} \]

\[ \beta \rightarrow -1 \Rightarrow \approx \text{UNIFORM} \]

FOR \( \beta > 0 \), TAILS ARE HEAVIER

FOR \( \beta < 0 \), TAILS DIE OFF MORE RAPIDLY

\[ \beta = -\frac{1}{2} \Rightarrow p(w) \propto e^{-c(\beta)(\frac{w}{\sigma})^{-4}} \]
Example: DC level in i.i.d. non-Gaussian noise

\[ H_0 : x(n) = \text{win} \quad n = 0, 1, \ldots, N-1 \]
\[ H_1 : x(n) = A + \text{win} \quad n = 0, 1, \ldots, N-1 \]

**Known**

\( \text{WIN's are i.i.d. with known PDF} \ p(\text{win}) \)

**NP decides** \( H_0 \) if

\[ L(x) = \frac{p(x; H_0)}{p(x; H_1)} > \theta \]

Due to i.i.d.

\[ L(x) = \prod_{n=0}^{N-1} p(x(n); H_0) \]

\[ \prod_{n=0}^{N-1} p(x(n); H_0) \]
\[ \frac{n-1}{n} \prod_{i=0}^{n-1} p(x_i) - A \]

\[ \frac{n}{n} \prod_{i=0}^{n-1} p(x_i) \]

DETERMINE \( H_1 \) IF

\[ \ln L(x) = \sum_{n=0}^{n-1} \frac{\ln p(x_i | x_{i-1})}{p(x_i)} > \ln \alpha = \gamma \]

LET \( g(x) = \ln \frac{p(x | A)}{p(x)} \)

DETERMINE \( H_1 \) IF

\[ \sum_{n=0}^{n-1} g(x(n)) > \gamma \]

CLEARLY \( g(x) \) DEPENDS ON NONE PDF.

1) GAUSSIAN

\[ g(x) = \ln \frac{1}{\sqrt{2\pi} \sigma^2} e^{-\frac{1}{2\sigma^2}(x-A)^2} \]

\[ = -\frac{1}{2\sigma^2} (x-A)^2 - \frac{1}{2\sigma^2} x^2 \]

\[ = -\frac{1}{2\sigma^2} (-2A + A^2) \]
\[ g(x) = \ln \left( \frac{1}{\sqrt{2\pi 0^2}} e^{-\frac{1}{2} \left( \frac{x - \mu}{\sqrt{0^2}} \right)^2} \right) \]

\[ = \sqrt{\frac{2}{0^2}} \left( 1x - \frac{1}{\sqrt{0^2}} x - \mu \right) \]

Note: The graph shows a logarithmic function with parameters \( A > 0 \).

\( g(\cdot) \) is now nonlinear.

A more intuitive detector results from letting \( y(n) = x(n) - \mu \).

\[ E(y(n) ; \theta_0) = -\frac{A}{N} \]

\[ E(y(n) ; \theta_1) = \frac{A}{N} \]

Symmetrized version.
\[ \sum g(x_i) > x' \]
\[ \Rightarrow \sum_{n=0}^{N-1} g(y(n) + A/2) > x' \]
\[ \text{or} \quad \sum_{n=0}^{N-1} h(y(n)) > x' \]

WHERE \[ h(y) = g(y + A/2) \]

\[ = \ln \frac{p(y - A/2)}{p(y + A/2)} \]

![Diagram of function h(y) and PDF](image)

**DETECTOR FIRST SUBTRACTS A/2 FROM DATA, THEN CLIPS IT, AND THEN SUMS.**

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X(n) + Y(n) -> h(y) -> \sum_{n=0}^{N-1} h(y) -> N/2 -> > x' -> y_1
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< x' -> y_0
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A/2
CLIPPER ELIMINATES NOISE SPIKES.

Note also that Clipper is memoryless (independence assumption) nonlinear transformation.

In general, for a signal as \( x(n) \),
we decide \( H_1 \) if

\[
\frac{1}{N} \sum_{n=0}^{N-1} g_n(x_1(n)) > \theta'
\]

where \( g_n(x) = \ln \frac{p(x - a_1/n)}{p(x)} \)

Let \( y(n) = x(n) - a_1/n \),

decide \( H_1 \) if

\[
\frac{1}{N} \sum_{n=0}^{N-1} h_n(y(n)) > \theta'
\]

where \( h_n(y) = \ln \frac{p(y - a_1/n)}{p(y + a_1/n)} \).
$h_0(y) = \ln \frac{p(y - A \sin(\omega t))}{p(y + A \sin(\omega t))}$

Note: If $p(w)$ is even, $h_0(y)$ will be odd ($h_0(-y) = -h_0(y)$).

Detection Performance

Difficult due to nonlinearity.

Use approximation valid for weak signals ($A \to 0$ and $n \to \infty$).

Assume signal is $A \sin(\omega t)$, where $A > 0$, what is $N_0$ detector and its performance as $A \to 0$?

Decide $H_1$ if $\sum_{n=0}^{N-1} g_n(x(n)) > t'$.
\[ g_n(x) = \sum_{n=0}^{\infty} p(x - A\sin(A)) \]

as \( A \to 0 \), nonlinearity can be linearized about \( A = 0 \)

\[ g_n(x) \approx g_n(x) \bigg|_{A=0} + \left. \frac{dg_n(x)}{dA} \right|_{A=0} A 
= 0 + \frac{dp(x - A\sin(A))}{p(x - A\sin(A))} \bigg|_{A=0} A 
\]

\[ = \frac{dp(w)}{A p(w)} \bigg|_{w = x - A\sin(A), A=0} \]

\[ = \frac{dp(x)}{A p(x)} \]

Decide \( H1.1 \) if

\[ \sum_{n=0}^{\infty} g_n(x101) \approx \sum_{n=0}^{\infty} -\frac{dp(x101)}{dx(101)} \]

or since \( A > 0 \)

\[ T(x) = \sum_{n=0}^{\infty} \frac{dp(x101)}{dx(101)} \]

\[ s(101) > 1 \]

\[ p(x101) \]
WHAT DO WE GET FOR GAUSSIAN NOISE?

NON-LINEARITY IS NOW

\[ g(x) = - \frac{d p(x)}{dx} \left( - \frac{d \ln p(x)}{dx} \right) \]

THIS DETECTOR IS THE SMALL SIGNAL NO DETECTOR = CALLED LOCALLY OPTIMUM DETECTOR. IF A IS LARGE, WILL NOT BE OPTIMUM.
CAN NOW FIND PERFORMANCE

ASYMPTOTIC NP DETECTOR AS

A \to 0 \quad n \to \infty

SEE APPENDIX 10A (T(x) STILL
NONLINEAR IN DATA BUT CAN USE
CENTRAL LIMIT THEOREM SINCE

\frac{dP(x|\theta_0)}{dx|\theta_0} \text{ IF IID UNDER } \theta_0
\frac{P(x|\theta_0)}{P(x|\theta_i)} \text{ IIF IID UNDER } \theta_i

\text{ RESULT: }

T(x) = \sum_{n=0}^{N-1} \frac{dP(x|\theta_0)}{dx|\theta_0} = \sum_{n=0}^{N-1} g(x|\theta_0) w(n)

\sim \mathcal{N}(0, i(A) \sum_{n=0}^{N-1} \sigma^2(w(n)) \quad H_0

\mathcal{N} \left( A i(A) \sum_{n=0}^{N-1} \sigma^2(w(n)), i(A) \sum_{n=0}^{N-1} \sigma^2(w(n)) \right) \quad H_1

\text{ WHERE } i(A) = \int_{-\infty}^{\infty} \left( \frac{dP(w)}{dw} \right)^2 \frac{dP(w)}{P(w)}
\[ I'(A) = \text{Fisher Information for } A\text{ based on single sample.} \]

This just mean-shifted Gauss-Gauss problem

\[ \Rightarrow P_D = Q\left( Q'\left(P_{FA}\right) - 1\right) \]

Where

\[ d^2 = \left( \frac{E(T; H_1) - E(T; H_0)}{\text{VAR}(T; H_0)} \right)^2 \]

\[ = A^2 \hat{c}(A) \sum_{n=0}^{\infty} S^2(n) \]

Example: Gaussian PDF IID samples

Decide \( H_0 \) if

\[ Q(x) = \sum_{n=0}^{\infty} g(x|n) S(n) > \delta' \]

\[ g(x) = -\frac{d p(y)}{dx} \]

\[ \frac{p(x)}{p(x)} = -\frac{d \ln p(x)}{dx} \]
\[ p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}} \]

\[ q(x) = -\frac{d}{dx} \left(-\frac{1}{2\sigma^2} x^2\right) \]

\[ = \frac{x}{\sigma^2} \]

\[ T(x) = \sum_{n=0}^{\infty} \frac{1}{\sigma^2} x(n) \mathcal{S}(x) \]

**Usual Correlator**

(for this case \( NP = \text{weak signal} \ NP \))

\[ d^2 = A^2 i(A) \sum_{n=0}^{\infty} x(n)^2 \]

\[ i(A) = \int \left( \frac{d}{dw} \frac{d\rho(w)}{d\omega} \right)^2 \frac{d\omega}{p(w)} \]

\[ = \int \left( \frac{d}{dw} \frac{d\rho(w)}{d\omega} \right)^2 p(w) \, dw \]

\[ = E \left[ \left( \frac{d}{dw} \frac{d\rho(w)}{d\omega} \right)^2 \right] \]
\[ = E \left[ y^2(w) \right] \]
\[ = E (w^2 / \theta^2) = \frac{1}{\theta^2} \]
\[ d^2 = \frac{A^2 \sum s^2(n)}{\theta^2} = \frac{3}{\theta^2} \]

As expected.

Aside: \( i(A) \) called intrinsic accuracy of PDF, is the Fisher information for the parameter \( \mu \).

If \( p(x; \mu) = p_w(x - \mu) \),

Gaussian PDF has smallest \( i(A) \) for all PDFs with same variance \( \Rightarrow \) Gaussian PDF is worst case (smallest \( d^2 \)).

Example: non-Gaussian noise

Consider Laplacian noise and a dc level \( A > 0 \).
DECIDE \( H_1 \) IF.

\[
T(x) = \sum_{n=0}^{N-1} \frac{-d \frac{p(x|\omega_n)}{dx|\omega_n}}{p(x|\omega_n)} > \gamma
\]

or

\[
T(x) = \sum_{n=0}^{N-1} \frac{-d \frac{p(x|\omega_n)}{dx|\omega_n}}{p(x|\omega_n)} > \gamma
\]

But

\[
p(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-1)^2}
\]

\[
g(x) = -\frac{d}{dx} \frac{p(x)}{p(x)} = -\frac{d \ln p(x)}{dx}
\]

\[
= -\frac{d}{dx} \left( -\sqrt{2\pi} \right)
\]

\[= \sqrt{2\pi} \frac{d}{dx} \frac{1}{\sqrt{2\pi}} = \sqrt{2\pi} \sin \omega(x)
\]

\[
T(x) = \sqrt{2\pi} \sum_{n=0}^{N-1} \sin \omega(x|\omega_n)
\]
WEAK SIGNAL DETECTOR FOR
DC LEVEL \( A > 0 \) IN IID LAPLACIAN
NOISE; ADD SAMPLE SIGNS.

CALLED A SIGN DETECTOR.

\[
g(x)/\sqrt{2}\sigma\rightarrow\begin{cases} 1 & \text{if } A > 0 \\ -1 & \text{if } A < 0 \end{cases}
\]

A KNOWN - NP

A UNKNOWN - A > 0
WEAK SIGNAL NP

PERFORMANCE (AS \( A \rightarrow 0 \)) IS

\[
P_D = Q(\frac{1}{\sqrt{2}} \frac{P_{FA}}{\sigma^2})
\]

\[
\sigma^2 = A^2 \hat{\epsilon}(A) \Sigma^2(a)
= A^2 \hat{\epsilon}(A) N
\]

TO FIND \( \hat{\epsilon}(A) \):
\[ \begin{align*}
\hat{I}(A) &= E(g^2(w)) \\
&= E((\sqrt{2/\sigma^2} \cdot \text{sgn}(w))^2) \\
&= \frac{2}{16} \cdot E(1) = \frac{2}{16} \\
&= \text{TWICE THAT FOR GAUSSIAN NOISE} \\
\bar{d}_{\text{Lap}}^2 &= \frac{2NA^2}{\sigma^2} = 2 \bar{d}_{\text{gaussian}}^2
\end{align*} \]

**Deterministic signals with unknown parameters**

Rao test is easier to implement than GRT.

Consider

\[ \begin{align*}
H_0 : \quad x(n) &= w[n] & n = 0, 1, \ldots, N-1 \\
H_1 : \quad x[n] &= A\sin(n) + w[n] \\
&\quad \uparrow \\
&\quad i.i.d. \text{ non-Gaussian} \\
&\quad \text{unknown} \quad p(w[n])
\end{align*} \]

\[-\infty < A < \infty\]