VII. Switched-Capacitor Networks

1. Introduction

A) Problems in continuous time filters

Example 1st order LP

\[ V_i \rightarrow \frac{V_o}{i} = \frac{V_i}{(1 + \frac{1}{\omega p R C})} \]

\[ \omega p = \frac{1}{RC} \]

\[ H(j\omega) \]

\[ C \quad \omega_p = \frac{2\pi \cdot 10 kHz}{C = 1 \mu F} \]

\[ R = 16 \, M\Omega \]

Problems:
- \( R \) too big
- Absolute control of \( R \) and \( C \) required

B) Solution

Replace resistors by periodically charged and discharged capacitors

\[ V_i \rightarrow V_o \rightarrow V_i \]

\[ + AV \]
\( \Phi_1, \Phi_2 \) Nonoverlapping 2-phase clock signal

\[ \Phi_1 \quad \Phi_2 \]

Resistor-Capacitor Relationship: (ideal switches)

\[ \Delta Q: \text{Transferred Charge per Cycle} \]

\[ \Delta Q = C (V_1 - V_2) \quad \frac{i}{T} \]

\[ \frac{\Delta Q}{T} = I = \frac{C}{T} \Delta V \quad \text{where } \Delta V = V_1 - V_2 \]

Compare with Ohm's Law:

\[ I = \frac{1}{R} \Delta V \]

\[ \Rightarrow \quad \frac{1}{R_{\text{eq}}} = \frac{T}{C} = \frac{1}{C f_0} \]

Note: The average current \( \bar{I} \) is not a quantity that can physically be measured; it is the equivalent current that transfers the charge \( \Delta Q \) in time \( T \) from node 1 to node 2 with \( V_1 - V_2 = \Delta V \).
If we assume ideal switches ($\tau_{on} = 0$), the charge transfer is realized by a train of delta functions.

If $\tau_{on}$ is included in the analysis, the charge transfer is realized by a train of exponentially damped functions.

**Example:** 1st order LP

\[
H(s) = \frac{1}{(1 + sR_{eq}C_2)} = \frac{1}{(1 + s \cdot \frac{C_2}{C_1})(1 + s/w_p)}
\]

\[w_p = \frac{1}{\sqrt{\frac{C_1}{C_2}}} = \sqrt{\frac{C_1}{C_2}}\]

\[f_c = \frac{1}{2\pi w_p} = \frac{C_2}{2\pi \cdot 100kHz}\]

\[C_2 = 1 \mu F\]

\[f_c = 200 kHz\]

\[C_1 = C_2 \cdot \frac{\frac{1}{w_p^2}}{f_c} = C_2 \cdot \frac{2\pi^2}{200kHz} = 0.514 \mu F\]
Advantages - Requires control of matrices of C only
- Small area (small C spread)
- Frequency scaling is possible with clock

Disadvantages - Requires continuous time anti-aliasing filter
- Requires "quartz stable" nonoverlapping 2-phase clock

2. CMOS Implementation

A) Switches: composite CMOS switch

\[ \text{Ron (kΩ)} \]

Advantages - MOSFETs exhibit no DC offset
- Extremely small leakage currents in off-state
- Ron variation acceptable over full signal swing
- 1st order elimination of clock feedthrough
B) Nonoverlapping 2-phase clock

Example with NOR Gates

\[ Q_1 = \overline{C_1 + \Phi_2} \]
\[ Q_2 = \overline{C_1 + Q_1} \]

\[ C_1 \]

\[ \Phi_1 \]

\[ \Phi_2 \]

\[ T \]

\[ t \]

\[ \frac{T_1}{T} = \frac{1}{2} \] 
\[ T_2 = \frac{1}{2} T - N t_d \]

\[ T = t + T_d \]

\[ T = \frac{1}{2} T - 2 t_d \]

**Notes:**
- If the delay of a single gate is too small, the above scheme can be modified to yield a longer nonoverlapping time by including an even number of inverters in the crossed-over feedbacks.

\[ T_{min} = (1 + N) t_d \]
\[ T_1 = \frac{1}{2} T - (2 + N) t_d \]
\[ T_2 = \frac{1}{2} T - N t_d \]

To reduce the clock feedthrough effect, it is also important to realize symmetrical edges for \( Q_1 \) and \( Q_2 \).
c) Capacitors

Undercut insensitive Structure

\[
\text{Uniform undercut (1.2 = 0.5 \mu)}
\]

\[
C_{oc} = C_{ox} \cdot (x - \Delta x)^2
\]

Ex: Realize 2 caps. with ratio 4

a) Minimum Area

b) Undercut insensitive + Gradient insensitive

\[
\frac{C_1}{C_2} = \frac{(x_0 - \Delta x)^2}{(x_0 - 2\Delta x)^2} = \frac{4 \left(1 - \frac{\Delta x}{x_0} + \frac{\Delta x^2}{2x_0^2}\right)}{(1 - 2\frac{\Delta x}{x_0} + \frac{\Delta x^2}{2x_0^2})}
\]

\[
\frac{C_1}{C_2} = \frac{4 (x_0 - \Delta x)^2}{(x_0 - 2\Delta x)^2} = 4
\]

\[
\approx 4 \left(1 - \frac{\Delta x}{x_0}\right) \left(1 + 2\frac{\Delta x}{x_0}\right) \Rightarrow 4 \left(1 + \frac{\Delta x}{x_0}\right)
\]

In general:

\[
\Rightarrow E_x \approx \frac{\Delta x}{x_0} \cdot 2 \left[1 - \sqrt{\frac{E_2}{E_1}}\right] \quad E_2 < E_1 \quad C_1 \approx C_2 \Rightarrow E_x = 0
\]
Capacitors with fractional ratios

\[ \frac{C_1}{C_2} = \pi, \quad 1 \leq \pi < 2 \]

\[ C_1 = C_{ox}(x_w-ax)(x_l-ax) \]
\[ C_2 = C_{ox}(x_o-ax)^2 \]

\[ \pi = \frac{C_1}{C_2} = \frac{(x_w-ax)(x_l-ax)}{(x_o-ax)^2} \]

Def: \[ w = \frac{x_w}{x_o} \quad \text{normalized width of } C_1 \]
\[ l = \frac{x_l}{x_o} \quad \text{normalized length of } C_1 \]
\[ e_u = \frac{ax}{x_o} \quad \text{undesired error of } C_2 \]

Thus:
\[ \pi = \frac{(w-e_u)(l-e_u)}{(1-e_u)^2} \quad (1) \]

Condition for zero undesired error:
\[ \left| \pi \right| - \pi \left( 4x_o \left( x_w - x_l + \frac{1}{4}ax^2 \right) \right) = \left( x_w + ax + x_l + \frac{1}{4}ax^2 \right) - \frac{1}{x_o^2} \]
\[ \left| \pi \right| - \pi \left( 2e_u^2 - e_u \right) = (w\bar{e}_u + l\bar{e}_u) - e_u^2 \]
\[ \left| \pi \right| - \left( 2 - e_u \right) + e_u = w + l \quad (2) \]

Solve for \( w \) and \( l \) in (1) + (2)

Solution:
\[ w = \pi \left( 1 - \sqrt{1 - \frac{1}{\pi}} \right) \]
\[ l = \pi \left( 1 + \sqrt{1 - \frac{1}{\pi}} \right) \]

\[ \pi = 1.5 \]
\[ w_{1.5} = 0.634 \]
\[ l_{1.5} = 3.166 \]
\[ x_o = 25 \mu m \]

\[ x_w = 15.85 \mu m \]
\[ x_l = 52.16 \mu m \]
5. SC Filter Design

A) The $z$-Transform

**Def**

\[ X(z) = \mathcal{Z}\{x(t)\} = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \]

Note: Very often $x(t) = 0$ for $t < 0$

Then

\[ X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} \quad \text{one-sided } z\text{-transform} \]

Recall, the Laplace transform of a discrete function

\[ X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) e^{-sn} \]

Relationship between $s$- and $z$-domain:

\[ z = e^{sT} \]

Inverse function:

\[ s = \frac{1}{T} \log_e |z| \quad \text{nontinear function} \]

i.e., polynomial in $s$ does not transform into polynomial in $z$
Mapping between $s$- and $z$-domain

a) Matched $z$-transform (MZT)

Poles and zeros in $s$ are exactly mapped into poles and zeros in $z$.

\[
(s + p) \rightarrow \frac{1}{T} \left(1 - e^{-Tz^{-1}}\right)
\]

b) Bilinear $z$-Transform (BDT)

The nonlinear function $s = \frac{1}{T} \ln[z]$ is approximated by a bilinear function:

\[
\text{Recall: } G(z) = 2 \left[\frac{s^{-1} + \frac{s^{-1}}{2}(\frac{s}{T})^2}{1 + \frac{s}{T}}\right] \\
\text{Approximation: } s \rightarrow \frac{2}{T} \left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)
\]

Frequency warping:

\[
s = j\omega \rightarrow \frac{2}{\pi} \tan\left(\frac{\omega}{2}\right)
\]

c) Lossless Discrete Int. $z$-transform (LDI)

The nonlinear function $s = \frac{1}{T} \ln[z]$ is approximated by a linear function:

\[
\text{Recall: } G(z) = \left(\frac{s^{-1} + \frac{s^{-1}}{2}(\frac{s}{T})^2}{1 + \frac{s}{T}}\right) \\
\text{Approximation: } s \rightarrow \frac{1}{T} \left(1 - z^{-1}\right)
\]

Frequency warping:

\[
s = j\omega \rightarrow \frac{2}{\pi} \sin\left(\frac{\omega}{2}\right)
\]
Mapping of the Frequency Axis

\[ \omega_{LDI} = \frac{1}{2} \sin \omega \frac{I}{2} \]
Continuos Int.

Forward-Diff. Int.

Backward-Diff. Int.

Lossles Discr. Int.

Bilinear Int.
3) The time-discrete integrator

### Continuous-time

\[
\frac{d}{dt} V_o(t) = -\frac{1}{\pi i C_2} V_i(t)
\]

\[
V_o(t) = -\frac{1}{\pi i C_2} \int V_i(t) \, dt
\]

\[
V_o(s) = -\frac{1}{\pi i C_2} \frac{1}{s} V_i(s)
\]

### Discrete-time

\[
V_o(n) - V_o(n-1) = -\frac{C_i}{C_2} V_i(n-\frac{1}{2})
\]

\[
\sum \left[ V_o(n) - V_o(n-1) \right] = -\frac{C_i}{C_2} \sum V_i(n-\frac{1}{2})
\]

\[
V_o(z) = -\frac{C_i}{C_2} \frac{z^{-\frac{1}{2}}}{1 - z^{-1}} V_i(z)
\]

#### Realized, s to z domain mapping

\[
s \rightarrow \frac{1}{T} \frac{1 - z^{-1}}{z^{-\frac{1}{2}}} \quad \text{LDI z-transform}
\]

#### Integrator constant

\[
\frac{1}{\pi i C_2} \rightarrow \frac{1}{T} \frac{C_i}{C_2}
\]

\[
\omega_i \rightarrow \frac{C_i}{C_2}
\]
The parasitic insensitive integrator

Inverting

\[
\begin{array}{c}
\text{C}_{p1}\quad \text{K}C\quad \text{C}_{p2}\quad \text{C}\quad \text{C}_{p3}
\\
V_i\quad 2\quad 2\quad \text{C}_{p4}
\\
\text{+}
\\
V_i\quad 2\quad 2\quad \text{C}_{p4}
\\
\text{+}
\\
\end{array}
\]

Noninverting

\[
\begin{array}{c}
\text{C}_{p1}\quad \text{K}C\quad \text{C}_{p2}\quad \text{C}\quad \text{C}_{p3}
\\
1\quad 1\quad 2\quad 2\quad \text{C}_{p4}
\\
\text{+}
\\
1\quad 1\quad 2\quad 2\quad \text{C}_{p4}
\\
\text{+}
\\
\end{array}
\]

we assume amplifier gain \( A_c \gg 1 \)

Influence of the different parasitics:

\( C_{p1} \): does not contribute to charge balance at ampl.

input node

\( C_{p2} \): switched between ground and virtual ground

\( \Rightarrow \) does not accumulate change

\( C_{p3} \): connected to virtual ground (amp. req. input)

\( \Rightarrow \) does not accumulate change

\( C_{p4} \): charged by amp. to \( V_o \), but does not contribute to charge balance at amp. input

\[
H^-(z) = -k \frac{z^{-1/2}}{(1 - z^{-1})} \quad H^+(z) = k \frac{z^{-1/2}}{(1 - z^{-1})}
\]

Rule to obtain parasitic insensitive circuits:

All effective capacitors in a SC network must be connected to nodes which are

grounded or virtually grounded (amplifier input)

or connected to a voltage source (amplifier output)
<table>
<thead>
<tr>
<th>BI-LINEAR INTEGRATOR</th>
<th>LOSSLESS DISCRETE INTEGRATOR</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Bilinear Integrator Diagram" /></td>
<td><img src="image2" alt="Lossless Discrete Integrator Diagram" /></td>
</tr>
</tbody>
</table>

**Bilinear Integrator**
- \( H(z) = \frac{1}{1+z^{-1}} \)

**Lossless Discrete Integrator**
- \( H(z) = \frac{z^{-1}}{1-z^{-1}} \)

\( k \) and \( k' \) are positive gains in the circuits.
1. Synthesis by Impedance Simulation

1.1 Resistance

\[ R = \frac{T}{C} = \frac{1}{C F_c} \]

1.2 Inductance

\[ L = \frac{T^2}{16C} \]
Synthesis by Impedance Simulation

Advantages:
- simple Synthesis Procedure
- requires few active Elements (Amplifiers)

Disadvantages:
- SC Circuits are adversely affected by Stray-Capacitances (top- and bottom-plate Parasitics).
- requires always a Predistortion Procedure (even for very high rates of Sampling- to Signal Frequency).
- may yield unstable SC Circuits.
2. Synthesis of SC Ladder Filters

General Procedure

i) Start with general doubly terminated LC-R ladder filter.

ii) Reduce given network to minimum reactance structure (by introducing controlled voltage-sources).

iii) Realize remaining reactive elements by SC integrators and controlled sources by SC summers.

iv) Calculate capacitor ratios according to the relationship between time continuous integrator and chosen SC realization.
Simulation of LC-II Ladder Filters

Example 3rd Order Lowpass Filter

cutoff frequency: \( f_0 = 1 \text{ kHz} \)
sampling rate: \( f_s = 16 \text{ kHz} \)

1. Rewriting of cutoff frequency for \( \text{LDI } z\)-transform

\[
\frac{f_0}{f_s} = \frac{f_s}{f_0} \sin \left( \frac{\pi \frac{f_0}{f_s}}{f_s} \right) = \frac{0.994 \text{ kHz}}{f_s}
\]

2. LC-II Prototype Filter

\[\begin{align*}
C_1 &= 1 \Omega \\
C_2 &= 1.621 \times 10^{-4} \text{ F} \\
L_3 &= 2.156 \times 10^{-4} \text{ H} \\
C_4 &= 2.917 \times 10^{-4} \text{ F}
\end{align*}\]

3. SFG in \( s\)-domain

\[
\begin{align*}
\text{NW equations:} \\
V_{c2} &= \frac{i}{C_2} \\
I_{c2} &= \frac{V_1 - V_{c2}}{C_2} \\
I_{c4} &= \frac{V_4 - V_{c4}}{C_4} \\
V_{c4} &= \frac{I_{c4}}{C_4} \\
I_{c5} &= \frac{i}{L_5} \\
V_{c5} &= V_{c2} - V_{c4} \\
I_{c5} &= \frac{1}{L_5} \\
V_4 &= \frac{1}{V_{c4}}
\end{align*}
\]

Impedance scaling by \( \frac{1}{C_1} \)
4. SFG in z-domain

Frequency scaling by $f_s$

5. Implementation of discrete-time integrator

equivalent integrator constant:

$$K_i = \begin{cases} \frac{f_s C_i}{G_1}, & i \text{ even} \\ f_s G_1 L_i, & i \text{ odd} \end{cases}$$

6. equivalent element values:

<table>
<thead>
<tr>
<th>passive</th>
<th>active SC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1 = 12^{-1}$</td>
<td>$G_{in} = G_1/10, = 1$</td>
</tr>
<tr>
<td>$C_2 = 1.621 \times 10^{-4} \text{ F}$</td>
<td>$K_2 = \frac{f_s}{G_1} C_2 = 2.595$</td>
</tr>
<tr>
<td>$L_3 = 2.156 \times 10^{-4} \text{ F}$</td>
<td>$K_3 = f_s G_1 L_3 = 5.417$</td>
</tr>
<tr>
<td>$C_4 = 2.417 \times 10^{-4} \text{ F}$</td>
<td>$K_6 = \frac{f_s}{G_1} C_4 = 3.867$</td>
</tr>
</tbody>
</table>
7. Implementation of resistive termination ($C_t$)

**Problem:** Input loop in $z$-domain SFG comprising $Q_1$ and $Q_2$ is not realizable as LTI equivalent circuit, since each loop delay must be an integer multiple of the sampling period.

**Solution 1:** Multiply $Q_1$ by $z^{-1/2}$

$Q_1 \rightarrow Q_1 z^{-1/2} = Q_1 (\cos \frac{\omega T}{2} + j \sin \frac{\omega T}{2})$

**Solution 2:** Multiply $Q_1$ by $z^{1/2}$

$Q_1 \rightarrow Q_1 z^{1/2} = Q_1 (\cos \frac{\omega T}{2} - j \sin \frac{\omega T}{2})$

**Note:** In both cases the passive real termination $Q_1$ becomes complex.

**Solution 3:** Combine solution 1 and 2 to eliminate a complex termination by multiplying $Q_1$ by $\frac{1}{2} (z^{1/2} + z^{-1/2})$

$Q_1 \rightarrow Q_1 \frac{1}{2} (z^{1/2} + z^{-1/2}) = Q_1 \cos \frac{\omega T}{2}$

(*bilinear transformation*)

**SC Implementation**

![Diagram of SC implementation](image)
3rd Order SC Ladder Filter

\[ K_2' = K_2 - G_1 / 2 \]

\[ K_4' = K_4 - G_2 / 2 \]
Example 2: 3rd order lowpass with finite zero

Filter specs:
- Passband edge: $f_1 = 1$ kHz
- Passband ripple: $< 0.5$ dB
- Stopband edge: $f_2 = 3$ kHz
- Stopband loss: $> 40$ dB
- Sampling rate: $f_s = 20$ kHz

1. R-LC Prototype Filter

![Diagram of R-LC filter circuit]

Normalized element values: $\left( C_n = C_i, f_n = f_s \right)$

<table>
<thead>
<tr>
<th>$C_1 = C_2 = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_i = 4$ pF</td>
</tr>
<tr>
<td>$L_2 = 5.2807$</td>
</tr>
<tr>
<td>$C_2 = 0.2611$</td>
</tr>
<tr>
<td>$C_3 = 4.8324$</td>
</tr>
</tbody>
</table>

2. Elimination of serial capacitor $C_2$

**NW equations:**

$\begin{align*}
I_{C_2} &= (V_{C_2} - V_{C_5}) C_2 \\
I_{C_1} &= I_{a_1} - I_{a_2} - I_{C_2} \\
V_{C_1} &= I_{C_1} \frac{1}{S C_1}
\end{align*}$

**Eq. (1)**

$\begin{align*}
I_{C_5} &= I_{L_2} + I_{C_2} - I_{a_2} \\
V_{C_5} &= I_{C_5} \frac{1}{S C_5}
\end{align*}$

**Eq. (2) and (3)**
Eliminating $I_{02}$ and plugging eq. (2) into (3) yields:

For C1 branches:
$$V_{C1} = (I_{01} - I_{02}) \frac{1}{C_{1}} - (V_{C1} - V_{C2}) \frac{C_2}{C_{1}}$$

$$V_{C2} = (I_{02} - I_{00}) \frac{1}{C_{2}} - (V_{C1} - V_{C2}) \frac{C_2}{C_{2}}$$

For C3 branches:
$$V_{C3} = (I_{03} - I_{02}) \frac{1}{C_{3}} + V_{S3} \frac{C_2}{C_{3} + C_{2}}$$

Finally, we substitute $(C_{1} + C_{2})$ by $C_{1}'$ and $(C_{2} + C_{3})$ by $C_{3}'$.

We can then write:

$$V_{C1} = (I_{01} - I_{02}) \frac{1}{C_{1}'} + V_{C3} \frac{C_2}{C_{1}'}$$

$$V_{C3} = (I_{02} - I_{00}) \frac{1}{C_{3}'} + V_{C1} \frac{C_2}{C_{3}'}$$

The equivalent circuit of this modified system is shown below:

![Equivalent Circuit Diagram]

where:

$$C_{1}' = C_{1} + C_{2}$$
$$C_{3}' = C_{2} + C_{3}$$
$$V_{1}' = V_{C3} \frac{C_2}{C_{1}'}$$
$$V_{3}' = V_{C1} \frac{C_2}{C_{3}'}$$

Note: This modified circuit contains only R, L, and C elements and can thus be realized by a simple active circuit.
4. **SC Implementation (LDE equivalent solution)**

Element correspondences: (see Fig. next page)

- \( C_{10} = C_1 - \frac{1}{2} C_2 \)
- \( C_{20} = L_2 \)
- \( C_{50} = C_3 - \frac{1}{2} C_2 \)
- \( C_{11} = C_1 \)
- \( C_{21} = L_1 \)
- \( C_{51} = 1 \)
- \( C_{12} = L_2 \)
- \( C_{22} = 1 \)
- \( C_{52} = C_2 \)
- \( C_{13} = C_1 \)
- \( C_{53} = 0 \)
- \( C_{14} = C_2 \)
3rd Order SC Ladder Filter with Finite Zero
5. Direct synthesis in the $z$-domain

SC Biqual
SEFG representation:

\[ V_0 = -F_1 F_3 z^{-1} - F_2 F_3 z^{-1} (1 - z^{-1}) + F_4 z^{-1} (1 - z^{-1}) - F_5 (1 - z^{-1})^2 \]

\[ V_c = (1 - z^{-1})^2 + B_1 F_3 z^{-1} + B_2 (1 - z^{-1}) \]

\[ \begin{align*}
V_0 &= \frac{F_5}{i + B_2} \left( 1 - 2 \frac{F_1 F_3 + F_2 F_3 - F_4}{F_5} + \frac{F_2 F_3 - F_4}{F_5} \right) \\
V_c &= \frac{F_5}{i + B_2} \left( 1 - 2 \frac{F_5}{1 + B_1} \right) + \frac{2}{i + B_2} \frac{F_5}{(1 + B_2)} 
\end{align*} \]  \hspace{1cm} (1)

Biquadratic Function in s:

\[ \frac{V_0}{V_c} = \frac{\left( \omega_e^2 + \frac{\omega_e}{\alpha} + \xi^2 \right)}{\left( \omega_p^2 + \frac{\omega_p}{\alpha} + \xi^2 \right)} \]

\[ \begin{align*}
V_0 &= -K_0 \left( 1 - 2 \frac{e^{i \omega_p T} \cos \omega_p T + \frac{i}{4} \omega_p^2 e^{i \omega_p T}}{1 - 2 \frac{e^{-i \omega_p T} \cos \omega_p T + \frac{i}{4} \omega_p^2 e^{-i \omega_p T}} \right) \\
V_c &= \frac{2 \omega_p^2}{\xi \omega_p} \lambda_2 \omega_p^2 \sqrt{1 - \frac{i}{4} \omega_p^2} 
\end{align*} \]  \hspace{1cm} (2)

where \( \lambda_2 = \frac{\omega_p}{2 \alpha_p} \), \( \omega_p = \omega_p \sqrt{1 - \frac{i}{4} \omega_p^2} \), \( T \) clock cycle

\[ \begin{align*}
\lambda_2 &= \frac{\omega_e}{2 \alpha_e} \\
\omega_e &= \omega_e \sqrt{1 - \frac{i}{4} \omega_e^2} 
\end{align*} \]
compare coefficients of equal powers in (1) and (2)

\[
\frac{F_5}{\frac{1}{1+\alpha^2}} = K_0
\]

\[
1 - \frac{F_1 F_3 + F_2 F_5 - F_4}{2 F_5} = e^{\frac{\alpha^2}{2} \cos \omega_p T}
\]

5 equations

7 unknowns

\[
1 - \frac{F_2 F_5 - F_4}{F_5} = e^{\frac{\alpha^2}{2} T}
\]

\[
1 - \frac{F_1 F_5 + \beta^2}{2 (1 + 18 \beta^2)} = e^{-\frac{\alpha^2}{2} \cos \omega_p T}
\]

\[
1 + 18 \beta^2 = e^{-2 \alpha^2 T}
\]

\Rightarrow 2 free parameters

For optimum dynamic (both amps exhibit equal max. swing)

\[
\beta_0 = F_5
\]

(better: \( F_1 = \frac{Q_p}{1+Q_p} \) for lowpass only)

minimum phase network

Nonminimum phase network

(zeros in left half plane)

(zeros in right half plane)

\[
Q_2 < 0 \Rightarrow F_4 = 0
\]

\[
Q_2 > 0 \Rightarrow F_2 = 0
\]
Example 1: Design a 2nd order lowpass with:

\[ f_p = 14 \text{kHz}; \ A_p = 1; \ \text{Unity Gain at } w = 0 \]
\[ f_s = 16 \text{kHz} \]

a) Transfer function in \( s \) and in \( z \):

\[
H(s) = -\frac{\omega_p^2}{\omega_p^2 + s\omega_p + s^2}
\]

\[
H(z) = \frac{z^2 - 2\cos(\omega_p T/2)z + 1}{1 - z^{-1} - 0.67525z^{-2}}
\]

b) Circuit Topology:

![Circuit Diagram]

\[
H(z) = \frac{P1P5}{(1 + z^2)(1 - z^{-1}L2 - \frac{2P1P5 + P2}{1 + z^2}z^{-2})}
\]
c) Coefficient comparison:

Note that the additional factor $z^{-1}$ in the $z$-domain transfer function numerator has no influence on the filter amplitude response, since it merely adds an excess linear phase

$$e^{-z^{-1}} = e^{-j\omega T}$$

Therefore:

\[
\begin{align*}
\frac{F_1 F_5}{1 + z^{-2}} &= 0.12591 \\
2 - \frac{F_1 F_5 + z^{-2}}{1 + z^{-2}} &= 1.54352 \\
\frac{1}{1 + z^{-2}} &= 0.67523
\end{align*}
\]

$$\beta_1 = F_5$$

(optimum dynamic)

not good for low-$Q$ filters

better $\beta_1 = \frac{Q_p}{1 + Q_p} F_5$ (lowpass only)

Solution:

\[
\begin{align*}
F_5 &= \beta_1 = 0.45182 \\
\beta_1 &= F_5 = 0.30531 \\
F_1 &= 0.45182 \\
\beta_2 &= 0.40098 \\
F_2 &= 0.48038 \\
F_3 &= 0.61063
\end{align*}
\]

Capacitors:

\[
\begin{align*}
C_{F1} &= 1 \\
(1) & \quad C_{F2} = 1.114 \\
(1) & \quad C_{F3} = 1.2588 \\
(1.258) & \quad C_{F1} = 1 \\
(1) & \quad C_{F2} = 2.516 \\
(2.078) & \quad C_{F3} = 2.516 \\
(2.078)
\end{align*}
\]
Example 3. Design the same lowpass filter by means of the bilinear $z$-transform.

a) Frequency prewarming:

$$f_p = \frac{f_s}{2\pi} \log \left( \frac{f_s}{f_p} \right) \approx 1.0\, \text{kHz}$$

b) Transfer function in $s$ and $z$-domain:

$$H(s) = \frac{\omega_p^2}{\omega_p^2 + s \frac{\omega_p}{Q} + s^2}$$

$$H(z) = \frac{\omega_p^2 T^2}{4(1+\omega_p T) + \omega_p^2 T^2 (1+T^{-1})^2}$$

$$= \frac{0.031944 (1+T^{-1})^2}{1 - T^{-1} - 1.85162 + T^{-2} + 0.67879}$$

c) Circuit Topology:

[Diagram of a circuit with components labeled]
Thus:

\[
\frac{F_5}{1 + \beta_2} = 0.051344
\]

\[
\frac{F_1 F_2}{F_5} = 4
\]

\[
2 - \frac{\beta_1 F_1 + \beta_2}{1 + \beta_2} = 1.55102
\]

\[
\frac{1}{1 + \beta_2} = 0.67479
\]

\[
\beta_1 \approx F_3
\]

Solution:

\[
\beta_1 = F_3 = 0.43387
\]

\[
\beta_2 = 0.47521
\]

\[
F_1 = 0.43387
\]

\[
F_5 = 0.047060
\]

Capacitors:

\[
C_{\beta_1} = 1
\]

\[
C_{\beta_2} = 10.055
\]

\[
C_{F_1} = 1
\]

\[
C_{F_3} = 9.220
\]

\[
C_{\beta_1} = 2.505
\]

\[
C_{F_5} = 1
\]

\[
C_2 = 21.250
\]
Design of a 2nd order SC Bandpass Filter

Filter Specs: 

- $f_p = 50$ kHz 
- $Q_p = 5$
- $f_s = 1$ MHz

Transfer Function:

$$H(z) = \frac{Z_2 Z_1}{(1 + Z_1 Z_2) (1 - Z^{-1} (1 - Z^{-1}))}$$

The above specs yield: $(M = 7)$

$$H(z) = -0.06283 \frac{Z^{-1} (1 - Z^{-1})}{(1 - 1.04422 Z^{-1} + 0.93910 Z^{-2})}$$

To balance the 2 amplifier output swings, we set $B_1 = F_3$

By comparing the coefficients of equal power in the above z-domain transfer functions, we obtain:

- $B_1 = F_1 = 0.51745$
- $B_2 = 0.06485$
- $F_2 = 0.19768$

Cap Values:

- $C_1 = 5.059$
- $C_2 = 15.420$
- $B_1 C_1 = 1.608$
- $F_2 C_1 = i$
- $F_3 C_2 = 4.901$
Example 3: Design a 2nd order bandpass with

\( f_p = 1 \text{kHz} \); \( Q_p = 5 \); Unity Gain in Passband
\( f_c = 15 \text{kHz} \)

a) \( z \)-domain transfer function (MFT equivalent)

\[
H_{np}(z) = -\frac{0.078540 (1 - z^{-1})}{1 - z^{-1} (1.77660 + z^{-2} 0.324465)}
\]

b) SC implementation

Select \( F_1 = F_4 = F_6 = 0 \).

\[
H_{np}(z) = -\frac{F_2 F_3 z^{-1/2} (1 - z^{-1})}{(1 + F_2) (1 - z^{-1} [2 - \frac{8F_3 + F_2}{(1 + F_2)} + z^{-2} \frac{1}{(1 + F_2)}])}
\]

c) Result from coefficient comparison:

\[
\begin{align*}
 I_1 &= 0.33993 \\
 I_2 &= 0.061707 \\
 F_2 &= 0.13638 \\
 F_3 &= 0.33993
\end{align*}
\]

\[
\begin{align*}
 C_1 &= 5.032 \\
 C_2 &= 12.253 \\
 C_{51} &= 2.036 \\
 C_{52} &= 1.0 \\
 C_F &= 2.361 \\
 C_F &= 4.835
\end{align*}
\]
Impulse Response of 2nd Order SC Bandpass/Highpass

SC Bandpass/Highpass Frequency Response ($f_s=200\text{kHz}$, $f_p=10\text{kHz}$, $Q_p=5$)

SC Bandpass/Highpass Phase Response ($f_s=200\text{kHz}$, $f_p=10\text{kHz}$, $Q_p=5$)
C-Spread Reduction Techniques

Example: Noninverting Integrator with T-section

\[ \text{Solution A} / \text{Solution B} \]

Difference integrator / integrator with T-section

Assumption: \( V_{\text{in}} \) only changes during phase 1

\[ \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{\beta \frac{z^{-\frac{1}{2}}}{(1 - z^{-1})}}{1 + 2\beta} \]

\[ \text{C-Spread} = \beta^{-1} \]

E.g. \( \beta = 0.01 \)

Select \( \beta = 0.1 \)

\[ \text{Spread} = 10 \]

E.g. \( \beta = 0.01 \) \( \therefore \) \( \beta = 0.105 \)

\[ \text{Spread} = 9.05 \]
Leapfrog Structure

Follow-the-Leader Feedback

Inverse Follow-the-Leader Feedback
4. Synthesis via Digital Filter Architecture

a) 2-opamp unit delay cell

b) Single opamp unit delay

5C Multiplier (digitally controlled) = mult. DAC
a) **Direct Form I Topology**

b) **Direct Form II Topology**
$\text{N-th order Circuit in Direct Form I Realization}$
$N$-th order Circuit in Direct Form II Realization
Influence of the amplifier finite gain

Discrete Integrator

\[ V_i \rightarrow kC_0 \rightarrow V_d \rightarrow -A_0V_d \]

\[ V_0 = -A_0V_d \]

Analysis

\[ \text{Charge at end of } Q_1 \text{ and Charge at end of } Q_2 \]

\[ Q_1 = Q_1(n + \frac{1}{2}) = kC_0V_i + C_0(V_{01} - V_{d1}) \]

\[ Q_2 = Q_2(n + 1) = -kC_0V_d + C_0(V_{02} - V_{d2}) \]

Charge conservation:

\[ Q_1 = Q_2 \quad 1 \rightarrow 2 \]

Replace \( V_d \) by \( -\frac{1}{A_0}V_0 \)

\[ Q_1 = kC_0V_i + C_0V_{01} \left(1 + \frac{1}{A_0}\right) \]

\[ Q_2 = \frac{k}{A_0}C_0V_{02} + C_0V_{02} \left(1 + \frac{1}{A_0}\right) \]

\[ V_0 \text{ does not change during } Q_1 \text{, i.e. } V_i(n + \frac{1}{2}) = V_0(n) \]

\[ Q_1(n + \frac{1}{2}) = kC_0V_i(n + \frac{1}{2}) + C_0V_{01}(n) \left(1 + \frac{1}{A_0}\right) \]

\[ Q_2(n + 1) = \frac{k}{A_0}C_0V_{02}(n + 1) + C_0V_{02}(n + 1) \left(1 + \frac{1}{A_0}\right) \]
\[
Q_1 = Q_2
\]

\[
\Rightarrow K \frac{1}{\beta} V_i (\alpha + \frac{1}{2}) + \frac{1}{\beta} V_o (\alpha) \left[ 1 + i \frac{1}{A_0} \right] = \frac{1}{\beta} V_o (\alpha + 1) \left[ 1 + i \frac{1}{A_0} + i \frac{1}{A_0} \right]
\]

Z-Transform

\[
K V_i (z) \equiv \frac{1}{\beta} V_i (\alpha + \frac{1}{2}) + \frac{1}{\beta} V_o (\alpha) \left[ 1 + i \frac{1}{A_0} \right] = \frac{1}{\beta} V_o (\alpha + 1) \left[ 1 + i \frac{1}{A_0} + i \frac{1}{A_0} \right]
\]

\[
\Rightarrow \frac{V_o (z)}{V_i (z)} = \frac{k \cdot z^{1/2}}{z \left[ 1 + i \frac{1}{A_0} + k \frac{1}{A_0} \right] - \left[ 1 + i \frac{1}{A_0} \right]} = \frac{k z^{1/2}}{(1 - z^{1/2})} \frac{(1 - z^{-1})}{(1 + \frac{1}{A_0} + k \frac{1}{A_0} - z^{-1} \left[ 1 + i \frac{1}{A_0} \right])}
\]

\[
\text{Closest \( z \): } \quad \text{Roots: } \quad \frac{1}{z} \quad \text{Coef.} = E(z)
\]

\[
E(z) \equiv (1 - \frac{1}{A_0} \left[ 1 + \frac{1}{A_0} \right]) \frac{(1 - z^{-1})}{1 - z^{-1} \left[ 1 - i \frac{1}{A_0} \right]}
\]

\[
E(\omega) \equiv (1 - \frac{1}{A_0} \left[ 1 + \frac{1}{A_0} \right]) \frac{2 \text{Re} \left[ e^{-j \omega T/2} \right]}{2 \text{Re} \left[ e^{-j \omega T/2} + \frac{k}{A_0} e^{-j \omega T/2} \right]}
\]

\[
E(\omega) \equiv \left( 1 - \frac{1}{A_0} \left[ 1 + \frac{1}{A_0} \right] \right) \frac{1 - j \frac{K}{A_0^2} \left[ \frac{1}{2A_0^2} \omega \right]}{1 - j \frac{K}{A_0^2} \left[ \frac{1}{2A_0^2} \omega \right]}
\]

\[
E(\omega) \approx 1 - \frac{1}{A_0} \left[ 1 + \frac{K}{A_0^2} \right] + j \frac{K}{A_0^2} \left[ \frac{1}{2A_0^2} \omega \right]
\]

\[
|E(\omega)| = \frac{1}{A_0} \left[ 1 + \frac{K}{A_0^2} \right]
\]

\[
\Theta_{A_0}(\omega) \approx \frac{L}{2A_0^2} \left[ \frac{K}{\omega \omega T/2} \right]
\]
Influence of the finite amplifier open-loop Gain $A_0$

\[ m_{A_0}(\omega) \approx -\frac{1}{A_0} \left( 1 + \frac{k}{2} \right) \]
\[ \theta_{A_0}(\omega) \approx \text{Arc tan} \left( \frac{1}{A_0} \frac{k}{2 \tan \omega T/2} \right) \]

Influence of the Stray-Capacitances

\[ m_s(\omega) \approx -\frac{1}{A_0} \left( \frac{C_{sd}}{C_0} + \frac{C_{st}}{2C_0} \right) \]
\[ \theta_s(\omega) \approx \text{Arc tan} \left( \frac{1}{A_0} \frac{C_{st}/C_0}{2 \tan \omega T/2} \right) \]
Influence of the amplifier unity-gain bandwidth and the switch ON resistances

SC Integrator

Applied Models for amplifier and switches

[Diagram of SC Integrator and amplifier block with Bode plot showing a -20 dB/dec slope]
SC Integrator during Phase 1

\[ \text{Diagram 1} \]

SC Integrator during Phase 2

\[ \text{Diagram 2} \]
Second-order SC Lowpass Circuit with finite Passband Zero