NOW: $E[s[n_0]x[n_0-\ell]] = E[s[n_0](s[n_0-\ell] + w(n_0-\ell])]$

$$= E[s[n_0]s[n_0-\ell]]$$
$$= r_s[\ell]$$

$$E[x[n_0-k]x[n_0-\ell]] = E[(s[n_0-k]+w[n_0-k])(s[n_0-\ell]+w[n_0-\ell])]$$

$$= E[s[n_0-k]s[n_0-\ell]] + E(w[n_0-k]w[n_0-\ell])$$

$$= r_s[\ell-k] + r_w[\ell-k]$$

$$\Rightarrow \quad r_s[\ell] = \sum_{k=-\infty}^{\infty} h[k](r_s[\ell-k] + r_w[\ell-k])$$

$$= \sum_{k=-\infty}^{\infty} h[k]r_s[\ell-k] + \sum_{k=-\infty}^{\infty} h[k]r_w[\ell-k]$$

$$= h[\ell] \star r_s[\ell] + h[\ell] \star r_w[\ell]$$

TAKING FOURIER TRANSFORMS

$$P_s(f) = H(f)P_s(f) + H(f)P_w(f)$$

$$\therefore \quad H_{opt}(f) = \frac{P_s(f)}{P_s(f)+P_w(f)}$$ (DOESN'T DEPEND ON n_0)

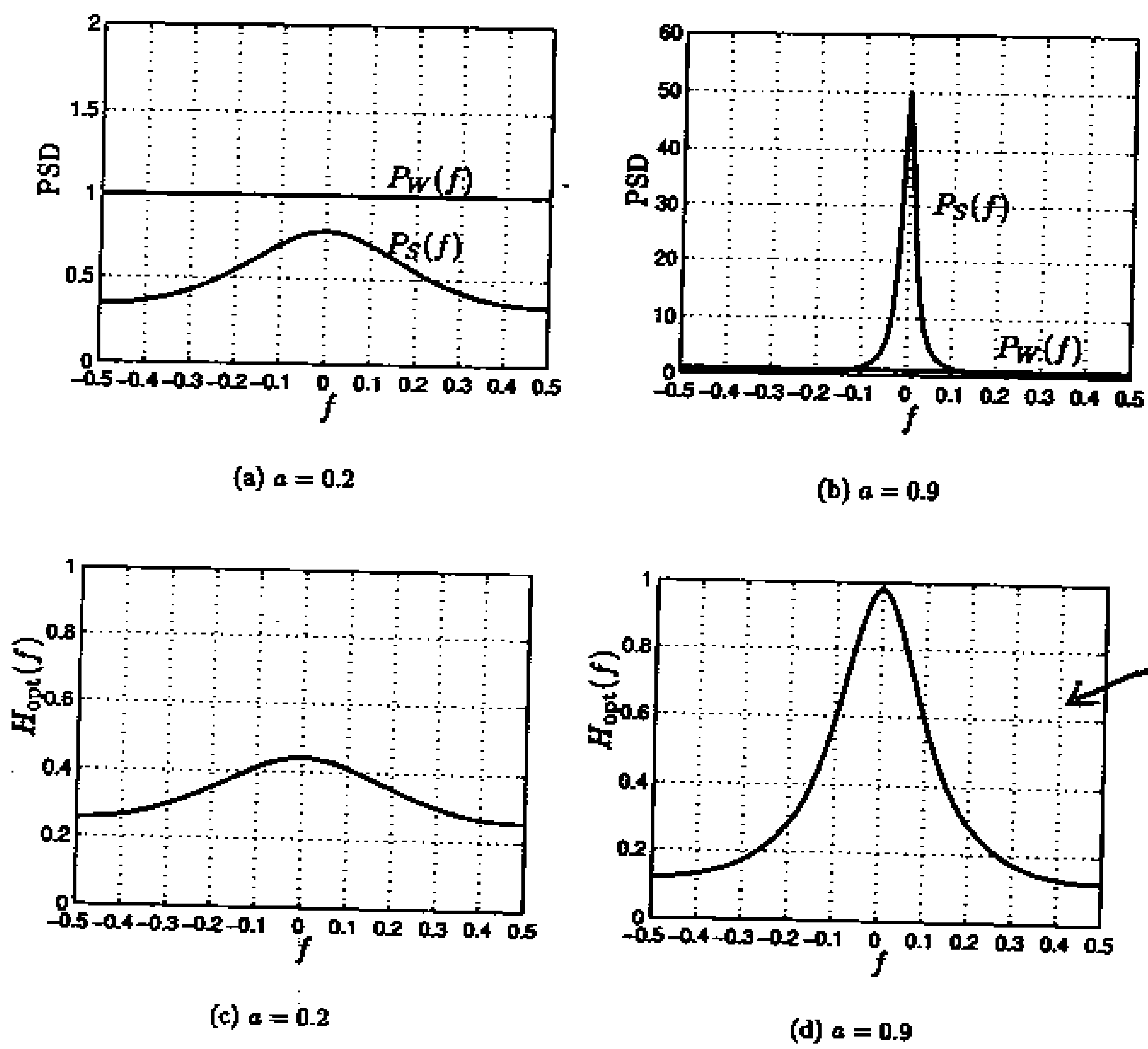
<u>EXAMPLE</u>: AR SIGNAL IN WHITE NOISE

$$P_v(f) = \frac{\sigma_v^2}{|1 - a e^{-j2\pi f}|^2}$$

$$P_W(f) = \sigma_W^2$$

$$H_{opt}(f) = \frac{\dfrac{\sigma_v^2}{|1 - a e^{-j2\pi f}|^2}}{\dfrac{\sigma_v^2}{|1 - a e^{-j2\pi f}|^2} + \sigma_W^2}$$

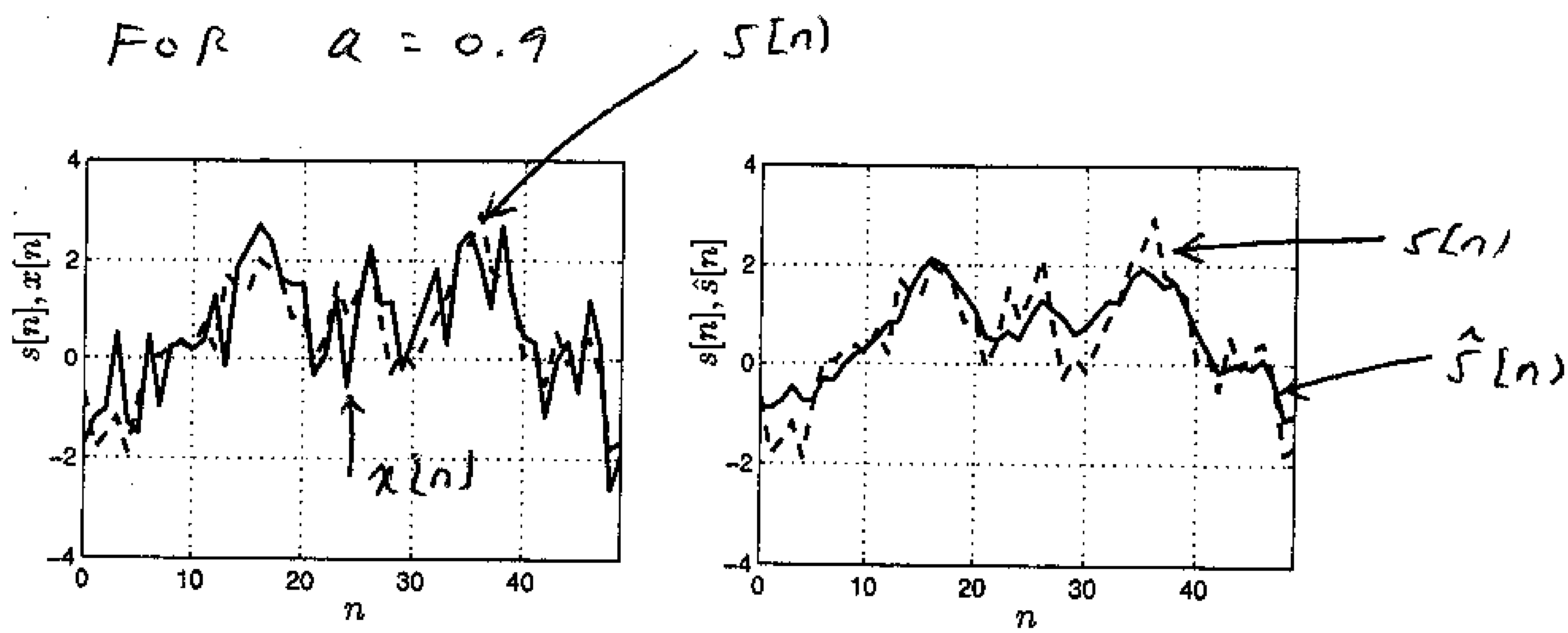
LET $a = \begin{cases} 0.2, \\ 0.9, \end{cases}$ $\sigma_v^2 = 0.5,$ $\sigma_W^2 = 1$



FOR WHICH FREQ. BAND DO WE GET GREATEST NOISE REDUCTION?

Figure 18.5: Power spectral densities of the signal and noise and corresponding frequency responses of Wiener smoother.

FOR $a = 0.9$



(a) True (dashed) and noisy (solid) signal

(b) True (dashed) and estimated (solid) signal

Figure 18.6: Example of Wiener smoother for additive noise corrupted AR signal. The true PSDs are shown in Figure 18.5b. In a) the true signal is shown as the dashed curve and the noisy signal as the solid curve and in b) the true signal is shown as the dashed curve and the Wiener smoothed signal estimate (using the Wiener smoother shown in Figure 18.5d) as the solid curve.

TO IMPLEMENT FILTER USE

$$\hat{s}[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} \underbrace{\frac{P_s(f)}{P_s(f)+\sigma_w^2}}_{H_{OPT}(f)} \underbrace{X_N(f)}_{\sum_{n=0}^{N-1} x[n] e^{-j2\pi f n}} e^{j2\pi f n}\, df$$

NOTE: THIS IS APPROXIMATE SINCE WE USE ONLY $\{x[0], x[1], \ldots, x[N-1]\}$ TO ESTIMATE $\{s[0], s[1], \ldots, s[N-1]\}$

SEE MATLAB CODE ON PAGE 615

CHAPTER 20 - GAUSSIAN RP

VERY IMPORTANT IN PRACTICE

1) PHYSICALLY MOTIVATED BY CLT
2) MATHEMATICALLY TRACTABLE
3) JOINT PDF OF SAMPLES $\sim N(\underline{\mu}, \underline{C})$
4) PDF SPECIFIED BY FIRST TWO MOMENTS
   a) CAN ESTIMATE THESE MOMENTS $\Rightarrow$ ESTIMATE OF $N(\underline{\mu}, \underline{C})$ PDF
   b) IF RP IS WSS $\Rightarrow$ STATIONARY
5) GAUSSIAN RP AT LINEAR FILTER INPUT $\Rightarrow$ GAUSSIAN RP AT OUTPUT

EXAMPLE : FISH COUNTING

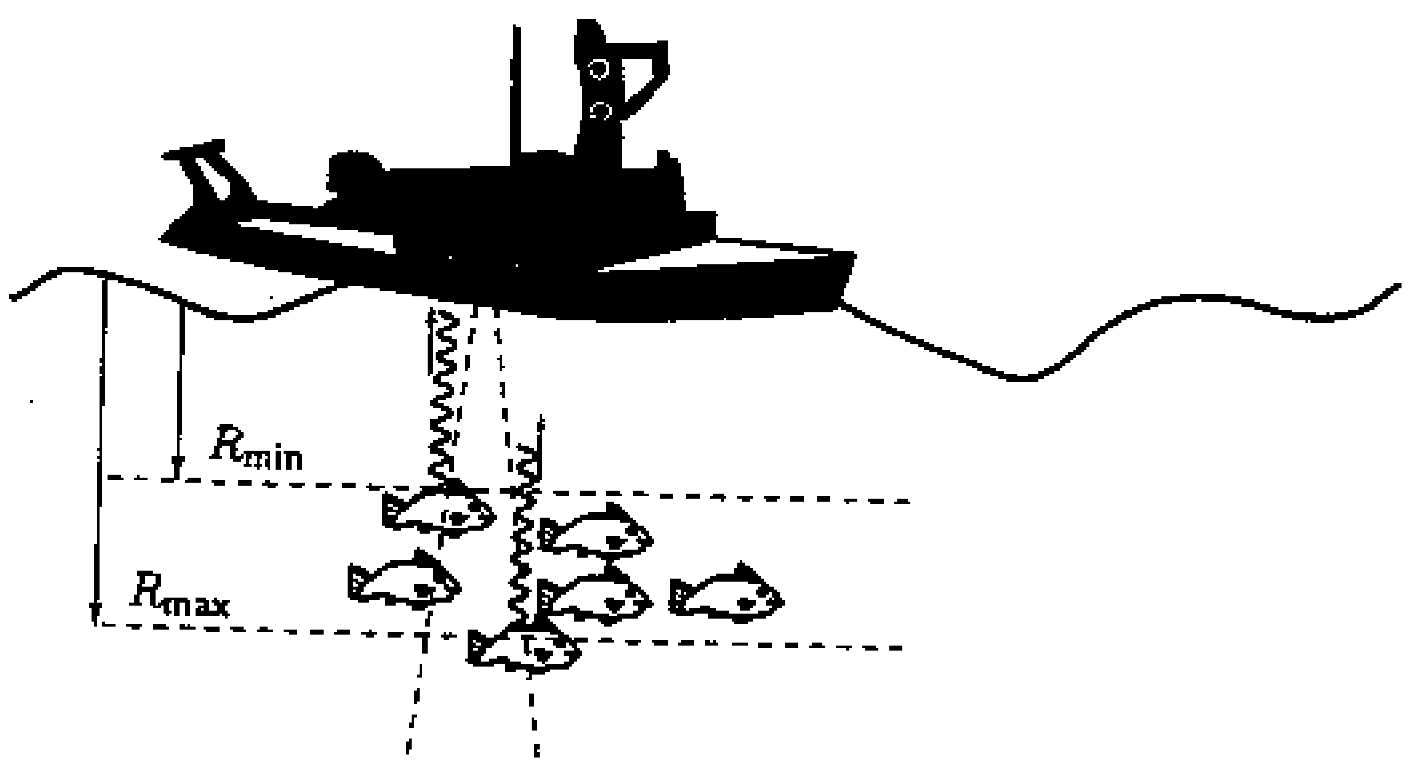


Figure 20.14: Fish counting by echo sonar.

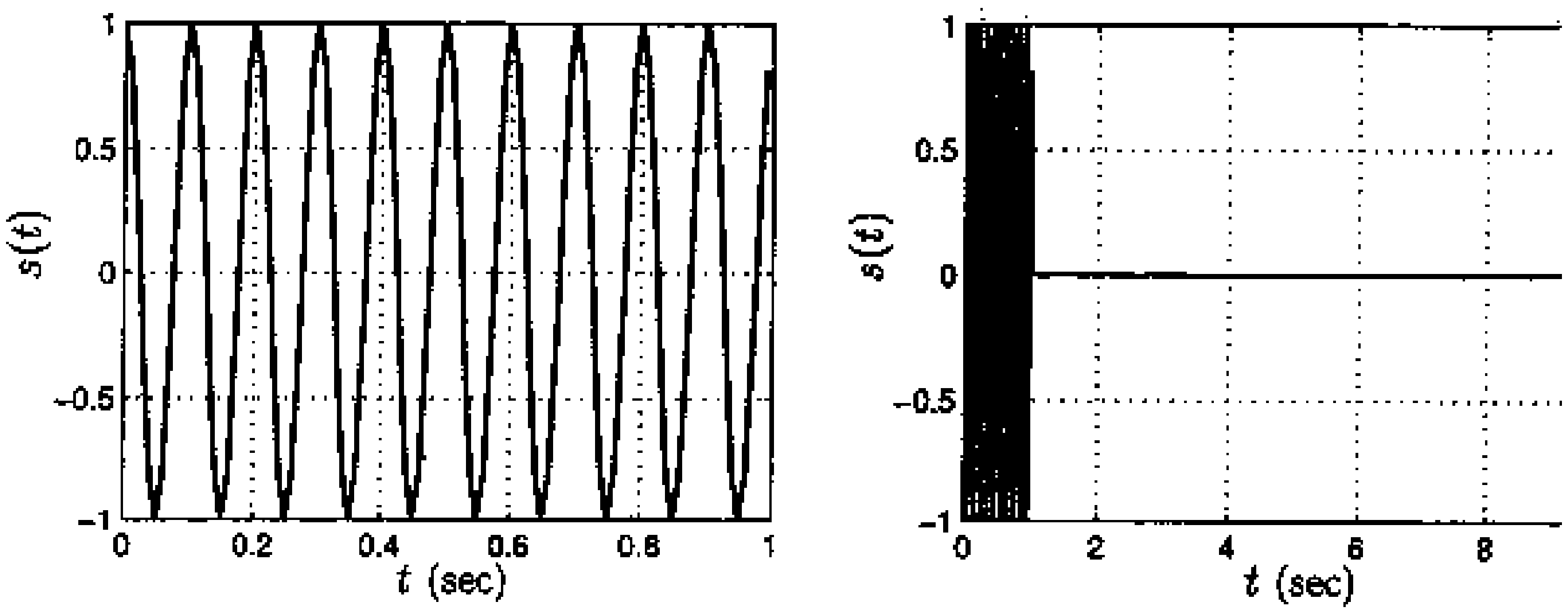
TRANSMIT BURST OF SINUSOID. AND ANALYZE RECEIVED WAVEFORM

TRANSMIT

$s(t) = \cos 2\pi F_0 t$

$F_0 = 10$ HZ $\quad 0 \le t \le T = 1$

(F_0, T CHOSEN FOR ILLUSTRATION ONLY!)



(a) Transmit pulse

(b) Transmit pulse shown in receive waveform observation window

Figure 20.1: Transmitted sinusoidal pulse.

ASSUME A LARGE NUMBER OF SCATTERED RETURNS

(⇒ CLT)

ASSUMES EACH RETURN IS

$A_i \cos(2\pi F_0 (t - \tau_i))$

$i = 1, 2, \ldots$

$A_i \sim U(0,1)$

$\tau_i = 0.1\, i + 0.1 U_i$

$U_i \sim U(-\frac{1}{2}, \frac{1}{2})$

ALL RVS ARE INDEPENDENT

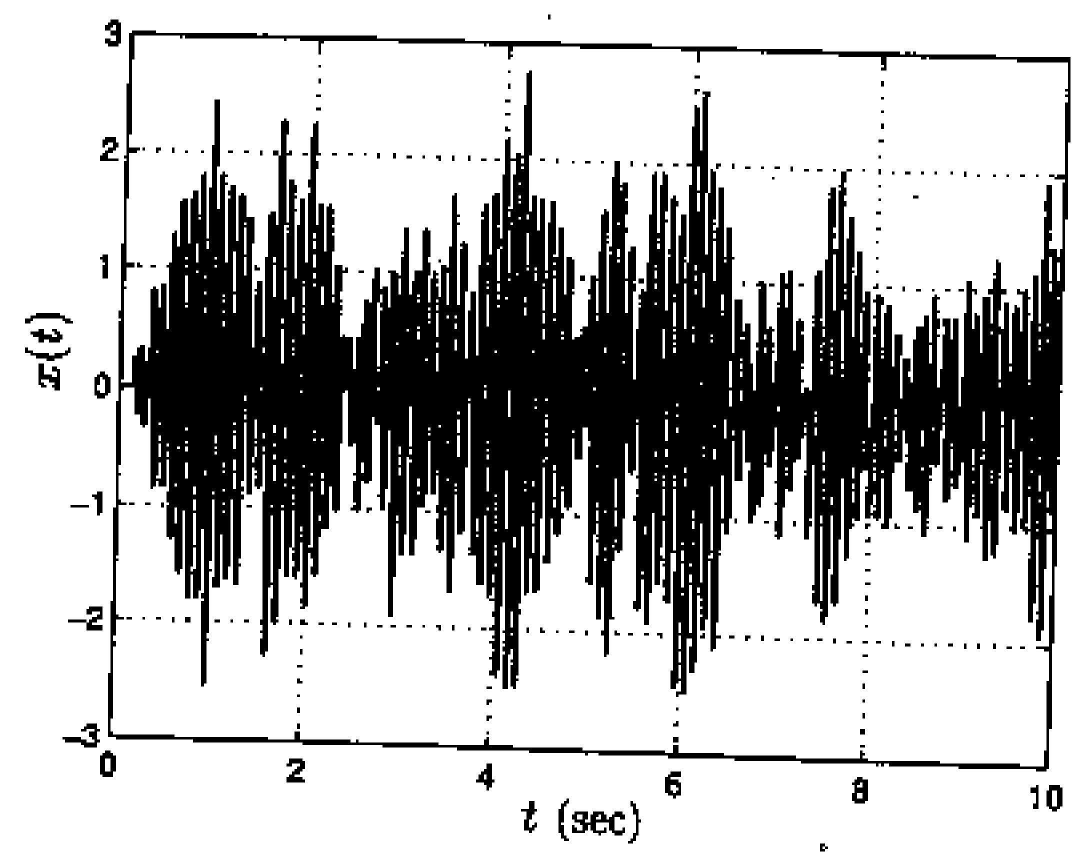
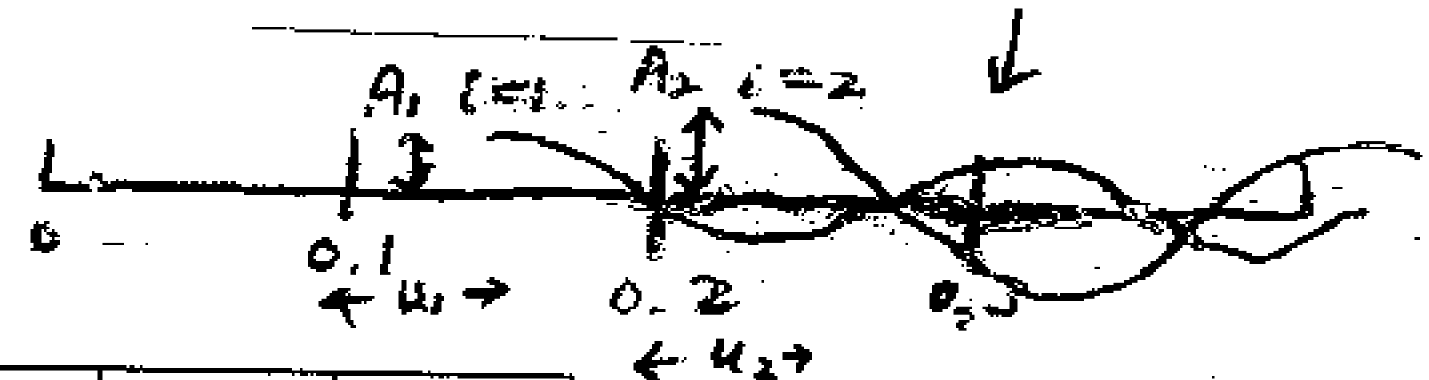


Figure 20.2: Received waveform consisting of many randomly overlapped and random amplitude echos.



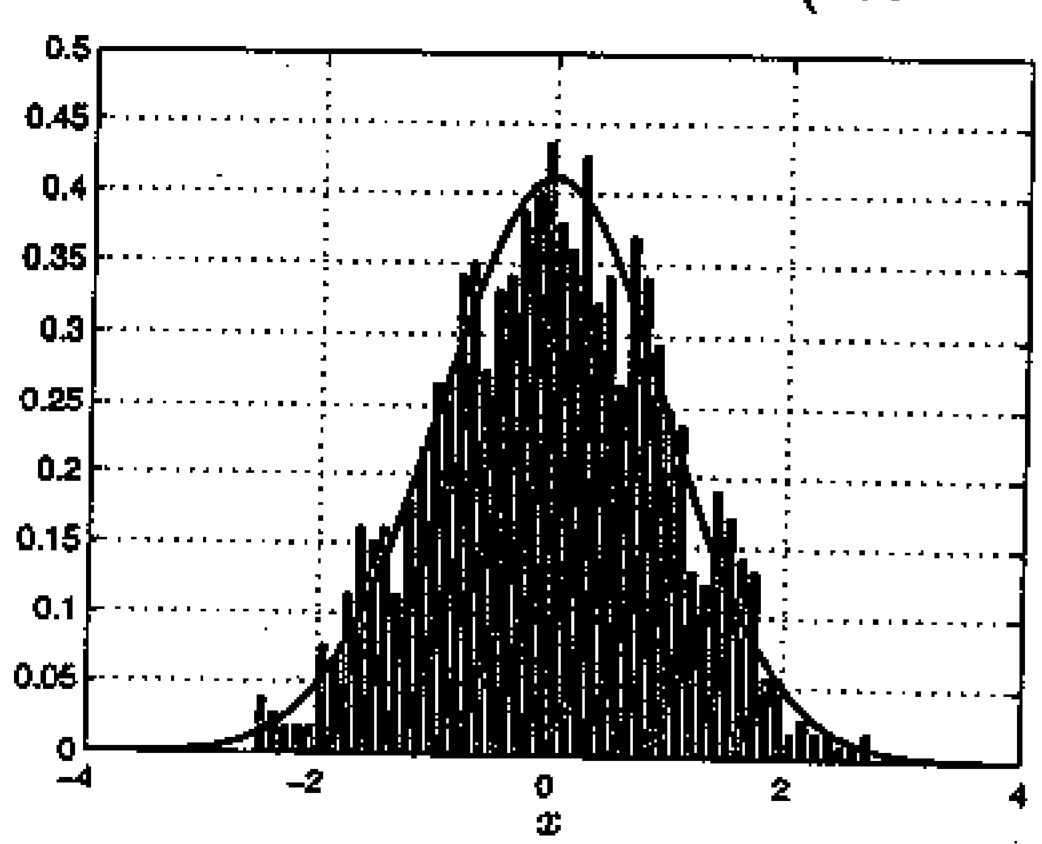


Figure 20.3: Marginal PDF of samples of received waveform shown in Figure 20.2 and Gaussian PDF fit.

DEFINITION OF GAUSSIAN RP

CONSIDER ONLY DISCRETE-TIME INITIALLY.

FIRST RECALL $\underline{x} = (x_1\ x_2 \ldots x_N)^T$
$\sim N(\underline{\mu}, \underline{C})$

PDF IS $$p_{\underline{x}}(\underline{x}) = \frac{1}{(2\pi)^{N/2} \, \mathrm{DET}^{1/2}(\underline{C})} \cdot e^{-\frac{1}{2}(\underline{x}-\underline{\mu})^T \underline{C}^{-1}(\underline{x}-\underline{\mu})}$$

WHERE $$\underline{\mu} = E_{\underline{x}}(\underline{x}) = \begin{bmatrix} E_{x_1}(x_1) \\ \vdots \\ E_{x_N}(x_N) \end{bmatrix} \quad N \times 1$$

$$\underline{C} = E_{\underline{x}}\left[(\underline{x} - E_{\underline{x}}(\underline{x}))(\underline{x} - E_{\underline{x}}(\underline{x}))^T\right] \quad N \times N$$

$$= \begin{bmatrix} \mathrm{VAR}(x_1) & \mathrm{COV}(x_1, x_2) & \ldots & \mathrm{COV}(x_1, x_N) \\ \mathrm{COV}(x_2, x_1) & \mathrm{VAR}(x_2) & \ldots & \mathrm{COV}(x_2, x_N) \\ & & \ddots & \\ \mathrm{COV}(x_N, x_1) & \mathrm{COV}(x_N, x_2) & \ldots & \mathrm{VAR}(x_N) \end{bmatrix}$$

PROPERTIES:

1) ONLY FIRST TWO MOMENTS REQUIRED, IE, $\underline{\mu}$ AND $\underline{C}$

2) IF ALL RVS ARE UNCORRELATED
$\Rightarrow \underline{C}$ IS DIAGONAL $\Rightarrow$ RVS ARE INDEPENDENT

3) $\underline{y} = \underline{G}\,\underline{x} \Rightarrow \underline{y} \sim N(\underline{G}\,\underline{\mu}, \underline{G}\,\underline{C}\,\underline{G}^T)$
$\uparrow$
$M \times N$
$M \leq N$

A RP IS DEFINED TO BE GAUSSIAN IF <u>ALL FINITE SETS OF SAMPLES HAVE A MULTIVARIATE GAUSSIAN PDF</u> (AS GIVEN ABOVE).

THIS SAYS THAT
$\underline{x} = [\,x[n_1]\; x[n_2] \ldots x[n_K]\,]^T$
HAS MULTIVARIATE GAUSSIAN PDF FOR ALL K AND ALL $\{n_1, n_2, \ldots, n_K\}$.

<u>EXAMPLE</u>: WGN

$\swarrow$ SEE EXAMPLE 16.6

DEFINED ORIGINALLY AS IID WITH $x[n] \sim N(0, \sigma^2)$. THIS SATISFIES DEFINITION SINCE

$$p_{\underline{x}}(\underline{x}) = \prod_{i=1}^{K} p_{x[n_i]}(x[n_i]) \quad \text{(INDEPENDENCE)}$$

$$= \prod_{i=1}^{K} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2\sigma^2} x^2[n_i]} \quad \text{(IDENTICALLY DIST. WITH } x[n] \sim N(0, \sigma^2)\text{)}$$

$$= \frac{1}{(2\pi\sigma^2)^{K/2}} e^{-\frac{1}{2\sigma^2}\underline{x}^T\underline{x}}$$

$$= \frac{1}{(2\pi)^{K/2}\, \mathrm{DET}^{1/2}(\sigma^2 \underline{I})} e^{-\frac{1}{2}\underline{x}^T(\sigma^2\underline{I})^{-1}\underline{x}}$$

OR $\underline{x} \sim N(\underline{0}, \sigma^2 \underline{I}) \Rightarrow$ WHITE GAUSSIAN

NOISE AS ORIGINALLY DEFINED IS A GAUSSIAN RP.

EXAMPLE. MA RP.

SEE EXAMPLE 16.7

RECALL DEFINED AS $X[n] = \frac{1}{2}(U[n] + U[n-1])$
WHERE $U[n]$ IS WGN WITH VARIANCE σ_U^2

TO SHOW IT IS A GAUSSIAN RP CONSIDER
$K=2$, $n_1 = 0$, $n_2 = 1$ (GENERAL CASE FOLLOWS IN SIMILAR MANNER)

$$\underbrace{\begin{bmatrix} X[0] \\ X[1] \end{bmatrix}}_{\underline{X}} = \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}}_{\underline{G}} \underbrace{\begin{bmatrix} U[-1] \\ U[0] \\ U[1] \end{bmatrix}}_{\underline{U}}$$

$$\Rightarrow \underline{X} \sim N(\underline{0}, \underline{G}\,\underline{C}_U \underline{G}^T) = N(\underline{0}, \sigma_U^2 \underline{G}\,\underline{G}^T)$$

MORE GENERALLY FOR $n_1 = n_0$, $n_2 = n_0 + 1$

$$\begin{bmatrix} x[n_0] \\ x[n_0+1] \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}}_{\underline{G}} \underbrace{\begin{bmatrix} u[n_0-1] \\ u[n_0] \\ u[n_0+1] \end{bmatrix}}_{\underline{u}}$$

SAME MULTIVARIATE GAUSSIAN PDF AS BEFORE. IN FACT, ANY LINEAR TRANSFORMATION OF $\underline{u}$ PRODUCES ANOTHER GAUSSIAN RANDOM VECTOR $\underline{x}$ $\Rightarrow$ $x[n]$ IS GAUSSIAN RP

RECALL THAT MA RP CAN BE VIEWED AS LSI SYSTEM

$$u[n] \longrightarrow \boxed{\tfrac{1}{2} + \tfrac{1}{2} z^{-1}} \longrightarrow x[n] = \tfrac{1}{2} u[n] + \tfrac{1}{2} u[n-1]$$

"SMOOTHING" FILTER

FOR EXAMPLE, IF FILTER IS INDEED A SMOOTHER $\Rightarrow$ $P[x[1]-x[0] > 1] < P[u[1]-u[0] > 1]$

BUT $u[1]-u[0] \sim N(0,2)$ $\qquad$ $u[n] \sim N(0,1)$ AND INDEPENDENT

$$\Rightarrow P(u[1]-u[0] > 1) = Q\left(\frac{1-0}{\sqrt{2}}\right) = Q(1/\sqrt{2}) = 0.2398$$

TO FIND $P[x[1]-x[0] > 1]$ LET $y = x[1]-x[0]$

$$y = \underbrace{[-1 \;\; 1]}_{\underline{A}} \underbrace{\begin{bmatrix} x[0] \\ x[1] \end{bmatrix}}_{\underline{x}} \sim N(\underline{A} E[\underline{x}], \underline{A} C_x \underline{A}^T)$$

$\underline{A}E(\underline{x}) = \underline{0}$ SINCE $x(n)$ IS ZERO MEAN

$$VAR(y) = \underline{A}\,\underline{C}_x\,\underline{A}^T = [-1 \quad 1]\,\underline{C}_x \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

BUT
$$\begin{bmatrix} x(0) \\ x(1) \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}}_{\underline{G}} \underbrace{\begin{bmatrix} v[-1] \\ v(0) \\ v(1) \end{bmatrix}}_{\underline{v}}$$

$$\underline{C}_x = \underline{G}\,\underline{C}_v\,\underline{G}^T = \underline{G}\,\underbrace{\sigma_v^2}_{=1}\,\underline{I}\,\underline{G}^T$$

$$= \underline{G}\,\underline{G}^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$VAR(y) = [-1 \quad 1] \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\begin{bmatrix} -1/4 \\ 1/4 \end{bmatrix}} = \frac{1}{2}$$

$$\Rightarrow \quad y \sim N(0, \tfrac{1}{2})$$

$$P\{x[1] - x[0] > 1\} = P(y > 1) = Q\left(\frac{1-0}{\sqrt{1/2}}\right) =$$
$$Q(\sqrt{2}) = 0.0786$$
$$< 0.2398$$
$$= P(v[1] - v[0] > 1)$$

<u>EXAMPLE</u>: DISCRETE-TIME WIENER PROCESS (BROWNIAN MOTION)

= RANDOM WALK WITH GAUSSIAN "STEPS"

$$x[n] = \sum_{i=0}^{n} v(i) \qquad n \geq 0$$

$v[n]$ IS WGN WITH VARIANCE σ_v^2

ANY SET OF SAMPLES ARE LINEAR TRANSFORMATION OF $v[i]$'S $\Rightarrow$ GAUSSIAN RP

$VAR(x[n]) = \underline{(n+1)\sigma_v^2} \Rightarrow$ NOT STATIONARY

PROPERTIES OF GAUSSIAN RP:

1) IF SAMPLES ARE UNCORRELATED, THEN THEY ARE INDEPENDENT. (WHY?)

2) IF GAUSSIAN RP IS WSS THEN IT IS ALSO <u>STATIONARY</u>.

<u>PROOF</u>: SHOW THAT WE GET SAME $N(\underline{\mu}, \underline{C})$ FOR SAMPLES $\{x[n_1], \dots, x[n_K]\}$ AND $\{x[n_1+n_0], \dots, x[n_K+n_0]\}$ FOR ALL K AND ALL $\{n_1, n_2, \dots, n_K\}$

PDF DEPENDS ONLY ON $\underline{\mu}$ AND $\underline{C}$

$$(\underline{\mu})_i = E[x[n_i+n_0]] \overset{DEF}{=} \mu_x(n_i+n_0) \overset{WSS}{=} \mu \overset{WSS}{=} \mu_x(n_i) \qquad i = 1, 2, \dots, K$$

$$(\underline{C})_{ij} = \text{cov}(x[n_i+n_0], x[n_j+n_0])$$

$$= E[x[n_i+n_0]\,x[n_j+n_0]] - E[x[n_i+n_0]] \cdot E[x[n_j+n_0]] \quad \text{(DEF)}$$

$$= r_x[n_j-n_i] - \mu^2 \quad \text{(WSS)}$$

$$= E[x[n_i]\,x[n_j]] - \mu^2 \quad \text{(DEF OF ACS)}$$

$$= E[x[n_i]\,x[n_j]] - E[x[n_i]]\,E[x[n_j]] \quad \text{(WSS)}$$

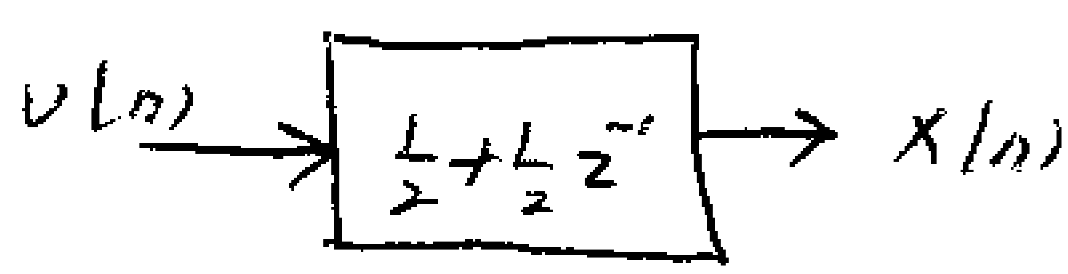
$$= \text{cov}(x[n_i], x[n_j]) \quad \text{FOR } i=1,2,\ldots,K;\; j=1,2,\ldots,K$$

LINEAR TRANSFORMATIONS

NEW GAUSSIAN RPS CAN BE DEFINED AS OUTPUTS OF LSI FILTERS WITH GAUSSIAN RP AS INPUT

RECALL MA PROCESS

$$x[n] = \tfrac{1}{2}(u[n] + u[n-1])$$



$\mathcal{H}(z)$ = SYSTEM FUNCTION

THIS IS A LINEAR TRANSFORMATION AS SHOWN PREVIOUSLY.

COULD ALSO HAVE

AR $\quad \mathcal{H}(z) = \dfrac{1}{1 - a z^{-1}} \qquad x[n] = a x[n-1] + \underbrace{u[n]}_{\text{WGN}}, \quad |a| < 1$

MA $\quad \mathcal{H}(z) = 1 - b z^{-1} \qquad x[n] = u[n] - b u[n-1]$

ARMA $\quad \mathcal{H}(z) = \dfrac{1 - b z^{-1}}{1 - a z^{-1}} \qquad x[n] = a x[n-1] + u[n] - b u[n-1], \quad |a| < 1$

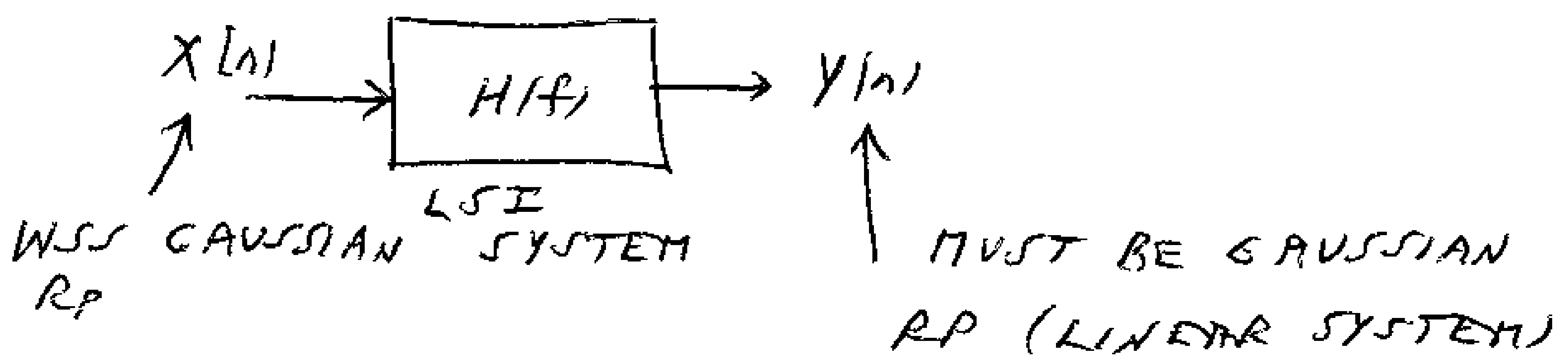
THESE ARE ALL GAUSSIAN RPS AND ARE USEFUL MODELS IN PRACTICE. MOST GENERAL IS ARMA (p, q)

$$\mathcal{H}(z) = \frac{1 - b[1] z^{-1} - b[2] z^{-2} - \cdots - b[q] z^{-q}}{1 - a[1] z^{-1} - a[2] z^{-2} - \cdots - a[p] z^{-p}}$$

$$x[n] = \sum_{k=1}^{p} a[k] x[n-k] + u[n] - \sum_{k=1}^{q} b[k] u[n-k]$$

$u[n]$ IS WGN WITH VARIANCE σ_u^2

IN GENERAL ASSUME $x[n]$ IS GAUSSIAN RP <u>AND</u> WSS WITH MEAN μ_x AND ACS OF $r_x[k]$.



ALSO, IF $x[n]$ IS WSS, THEN $y[n]$ IS WSS (THEOREM 18.3.1) <u>AND</u>

$$\mu_y = \mu_x H(0)$$
$$P_y(f) = |H(f)|^2 P_x(f)$$

(AND $x[n]$, $y[n]$ ARE ALSO STATIONARY)

<u>SUMMARY</u>: LINEAR SHIFT INVARIANT FILTERING OF WSS GAUSSIAN RP PRODUCES ANOTHER WSS GAUSSIAN RP BUT WITH A DIFFERENT MEAN AND PSD (ACS).

<u>EXAMPLE</u>: FOR RP MODELS, ARMA IS MOST GENERAL

$$\mu_x = \underset{\uparrow = 0}{\mu_u} H(0) = 0$$

$$P_x(f) = |H(f)|^2 P_u(f)$$
$$= |\mathcal{H}(e^{j2\pi f})|^2 \sigma_u^2$$
$$= \frac{\sigma_u^2 |1 - b[1]e^{-j2\pi f} \dots - b[q]e^{-j2\pi f q}|^2}{|1 - a[1]e^{-j2\pi f} - \dots - a[p]e^{-j2\pi f p}|^2}$$

EXAMPLE: DIFFERENCER - USED TO GET RID OF "DC LEVEL"

$$x[n] \longrightarrow \boxed{1 - z^{-1}} \longrightarrow y[n] = x[n] - x[n-1]$$

$x[n]$: WSS GAUSSIAN, μ_x, $r_x[k]$

$\mathcal{H}(z)$

QUESTION: WHAT IS PDF OF $\underline{y} = \begin{bmatrix} y[0] \\ y[1] \end{bmatrix}$?

$\underline{y}$ IS MULTIVARIATE GAUSSIAN - WHY?
NEED $\underline{\mu}_y$, $\underline{C}_y$ ONLY

BUT $y[n]$ IS WSS - WHY?

$$\Rightarrow \quad \overset{\text{WSS}}{\mu_y} = E[y[n]] = E[x[n] - x[n-1]]$$
$$= \mu_x[n] - \mu_x[n-1] \overset{\text{WSS}}{=} \underbrace{\mu_x - \mu_x}_{\text{"DC LEVEL"}} = 0$$

TO FIND $\underline{C}_y = \begin{bmatrix} E[y[0]y[0]] & E[y[0]y[1]] \\ E[y[1]y[0]] & E[y[1]y[1]] \end{bmatrix}$

$$= \begin{bmatrix} r_y[0] & r_y[1] \\ r_y[1] & r_y[0] \end{bmatrix} \quad \text{WSS}$$

BUT $\quad P_y(f) = |H(f)|^2 P_x(f)$

$$= H(f) H^*(f) P_x(f)$$
$$= (1 - e^{-j2\pi f})(1 - e^{j2\pi f}) P_x(f)$$
$$= 2P_x(f) - e^{j2\pi f} P_x(f) - e^{-j2\pi f} P_x(f)$$

TAKE INVERSE FOURIER TRANSFORM

$$r_y[k] = 2r_x[k] - r_x[k+1] - r_x[k-1]$$

$$\Rightarrow \quad \underline{C}_y = \begin{bmatrix} 2(r_x[0] - r_x[1]) & 2r_x[1] - r_x[2] - r_x[0] \\ & \end{bmatrix}$$

$$p_{y[0],y[1]}(y[0], y[1]) = \frac{1}{2\pi \, \mathrm{DET}^{1/2}(\underline{C}_y)} e^{-\frac{1}{2}\underline{y}^T \underline{C}_y^{-1} \underline{y}}$$

NONLINEAR TRANSFORMATIONS

IN GENERAL, IT IS HARD TO FIND PDF OF RP $y[n] = x[n] + \frac{1}{2}x^2[n]$ FOR $\underline{y} = [y[n_1] \; y[n_2] \ldots y[n_k]]^T$. IF $x[n]$ IS GAUSSIAN RP, WE CAN AT LEAST FIND <u>JOINT MOMENTS</u> OF $y[n]$.

<u>EXAMPLE</u>: $E[y[0]y[1]] = E[(x[0] + \frac{1}{2}x^2[0])(x[1] + \frac{1}{2}x^2[1])]$

$$= E[x[0]x[1]] + \tfrac{1}{2}E[x[0]x^2[1]] + \tfrac{1}{2}E[x^2[0]x[1]] + \tfrac{1}{4}E[x^2[0]x^2[1]]$$

$\Rightarrow$ NEED 2^{nd} ORDER, 3^{rd} ORDER, 4^{th} ORDER MOMENTS

BUT

$$E(x_1^{l_1} x_2^{l_2} \cdots x_N^{l_N}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} x_1^{l_1} x_2^{l_2} \cdots x_N^{l_N} \underbrace{p_{x_1,x_2,\ldots,x_N}(x_1,x_2,\ldots,x_N)}_{\text{MULTIVARIATE GAUSSIAN PDF = FUNCTION OF FIRST AND SECOND ORDER MOMENTS ONLY}} dx_1\, dx_2 \cdots dx_N$$

$\Rightarrow$ ALL JOINT MOMENTS DEPEND ON $\underline{\mu}$, $\underline{C}$ ONLY.

FOR PREVIOUS EXAMPLE NEED FORMULA FOR $E[x(i)x(j)x(k)]$ AND $E[x(i)x(j)x(k)x(l)]$ ($E[x(i)x(j)]$ GIVEN)

IMPORTANT RESULT: IF $\underline{x} = [x_1 x_2 x_3 x_4]^T \sim N(\underline{0}, \underline{C})$, THEN

$$E(x_1x_2x_3x_4) = E(x_1x_2)E(x_3x_4) + E(x_1x_3)E(x_2x_4) + E(x_1x_4)E(x_2x_3)$$

NOW ASSUME $x[n]$ IS A GAUSSIAN RP WITH ZERO MEAN

$$E[x(n_1)\,x(n_2)\,x(n_3)\,x(n_4)] =$$
$$E[x(n_1)\,x(n_2)]\;E[x(n_3)\,x(n_4)]$$
$$+\,E[x(n_1)\,x(n_3)]\;E[x(n_2)\,x(n_4)]$$
$$+\,E[x(n_1)\,x(n_4)]\;E[x(n_2)\,x(n_3)]$$

$\Rightarrow$ NEED ONLY KNOW SECOND ORDER MOMENTS

IN PARTICULAR, IF $x(n)$ IS ALSO WSS

$$E[x(n_1)\,x(n_2)\,x(n_3)\,x(n_4)] = r_x(n_2-n_1)\,r_x(n_4-n_3)$$
$$+\,r_x(n_3-n_1)\,r_x(n_4-n_2)$$
$$+\,r_x(n_4-n_1)\,r_x(n_3-n_2)$$

WHERE $r_x(k)$ IS KNOWN.

EXAMPLE : SQUARING OF WSS GAUSSIAN R.P.

$$\underset{\text{ZERO MEAN}}{x(n)} \longrightarrow \boxed{(\ )^2} \longrightarrow y(n) = x^2(n)$$

CLEARLY $y(n)$ IS NO LONGER GAUSSIAN RP - WHY?
IS IT AT LEAST WSS?

$$E[y(n)] = E[x^2(n)] = r_x(0) = \mu_y$$

$$E[y(n)\,y(n+k)] = E[x^2(n)\,x^2(n+k)]$$
$$= E[\underset{n_1}{x(n)}\;\underset{n_2}{x(n)}\;\underset{n_3}{x(n+k)}\;\underset{n_4}{x(n+k)}]$$

$$= r_X[0]\, r_X[0] + r_X[k]\, r_X[k] + r_X[k]\, r_X[k]$$

$$= r_X^2[0] + 2\, r_X^2[k]$$

NOT DEPENDENT ON n

$\Rightarrow$ $Y[n]$ IS WSS WITH

$\mu_Y = r_X[0]$

$r_Y[k] = r_X^2[0] + 2\, r_X^2[k]$

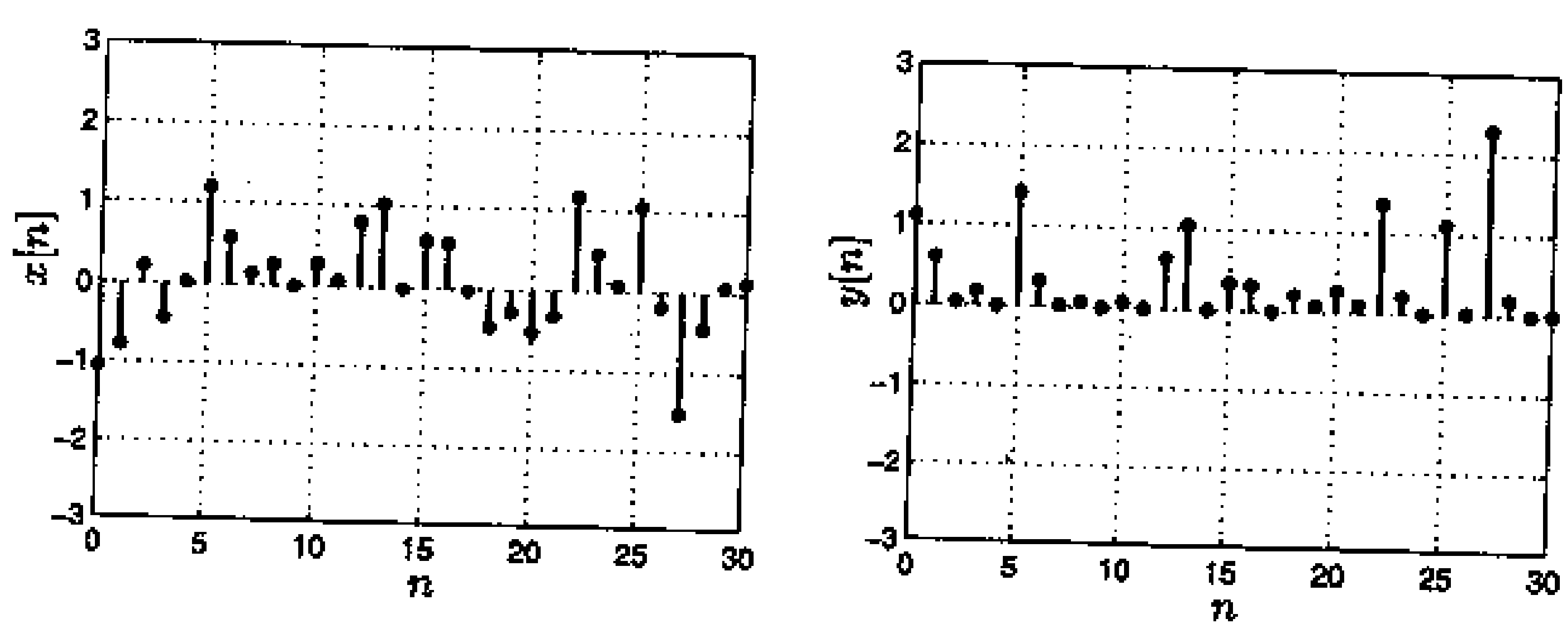
HOW IS PSD AFFECTED BY SQUARING?

$$P_Y(f) = \mathcal{F}\{ r_Y[k] \}$$

$$= r_X^2[0]\, \delta(f) + 2 \underbrace{P_X(f) \star P_X(f)}_{\int_{-\frac{1}{2}}^{\frac{1}{2}} P_X(\nu) P_X(f-\nu)\, d\nu}$$

NOW LET $X[n]$ BE MA RP

$X[n] = \frac{1}{2}(U[n] + U[n-1])$ $\quad \sigma_U^2 = 1$



(a) MA random process

(b) Squared MA random process

Figure 20.4: Typical realization of a Gaussian MA random process and its squared realization.

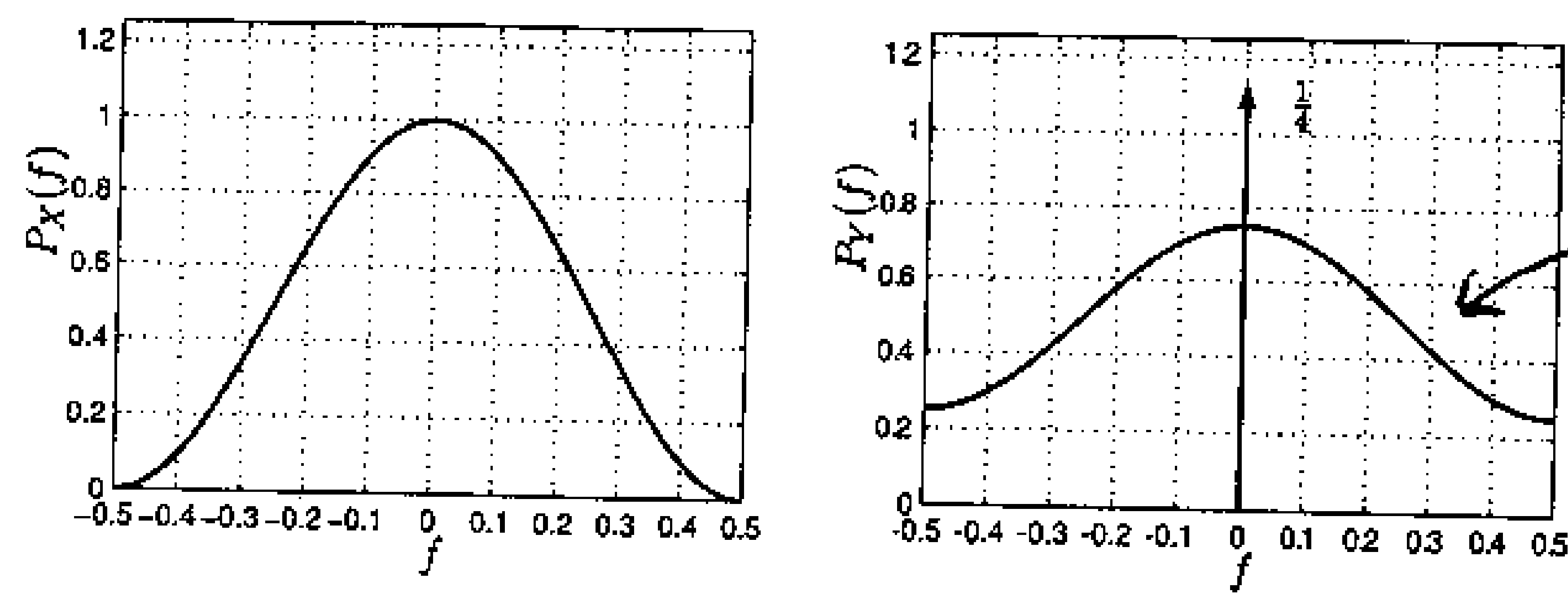
RECALL THAT FOR MA RP WITH $\sigma_u^2 = 1$

$$r_X[k] = \frac{1}{2}\delta[k] + \frac{1}{4}\delta[k+1] + \frac{1}{4}\delta[k-1]$$

$$\Rightarrow \quad \mu_Y = r_X[0] = \frac{1}{2}$$

$$\begin{aligned}
r_Y[k] &= r_X^2[0] + 2r_X^2[k] \\
&= 1/4 + 2\left(\frac{1}{2}\delta[k] + \frac{1}{4}\delta[k+1] + \frac{1}{4}\delta[k-1]\right)^2 \\
&= \frac{1}{4} + 2\left(\frac{1}{4}\delta[k] + \frac{1}{16}\delta[k+1] + \frac{1}{16}\delta[k-1]\right) \\
&= \frac{1}{4} + \frac{1}{2}\delta[k] + \frac{1}{8}\delta[k+1] + \frac{1}{8}\delta[k-1]
\end{aligned}$$

$$\begin{aligned}
P_Y(f) &= \frac{1}{4}\delta(f) + \frac{1}{2} + \frac{1}{8}e^{j2\pi f} + \frac{1}{8}e^{-j2\pi f} \\
&= \frac{1}{4}\delta(f) + \frac{1}{2} + \frac{1}{4}\cos 2\pi f \qquad |f| \le \frac{1}{2}
\end{aligned}$$



(a) MA random process

(b) Squared MA random process

PSD IS WIDENED DUE TO CONVOLUTION

Figure 20.5: PSDs of Gaussian MA random process and the squared random process.

CONTINUOUS-TIME GAUSSIAN RP

NOW HAVE $X(t) \quad -\infty < t < \infty$.

$X(t)$ IS DEFINED TO BE GAUSSIAN RP IF

$\underline{X} = [X(t_1) X(t_2) \ldots X(t_K)]^T$ HAS A

MULTIVARIATE GAUSSIAN PDF FOR ALL K

AND ALL $\{t_1, t_2, \ldots, t_K\}$