BUT $\Gamma(N) = (N-1)! \Rightarrow \Gamma(1) = 0! = 1$

$p_X(x) = \lambda e^{-\lambda x} \quad x \geq 0 \Rightarrow$ EXPONENTIAL

2) CHI-SQUARED (χ_N^2) WITH N DEGREES OF FREEDOM

$\alpha = N/2, \quad \lambda = \frac{1}{2}$

$$p_X(x) = \frac{\left(\frac{1}{2}\right)^{N/2}}{\Gamma(N/2)} x^{N/2-1} e^{-\frac{1}{2}x} \quad x \geq 0$$

$$= \frac{1}{2^{N/2}\Gamma(N/2)} x^{N/2-1} e^{-\frac{1}{2}x} \quad x \geq 0$$

χ_N^2 = SUM OF SQUARES OF "INDEPENDENT" $N(0,1)$S

3) ERLANG

$\alpha = N$

SEE BOOK (PDF FOR SUM OF INDEPENDENT EXP. RVS)

7) RAYLEIGH

$$p_X(x) = \begin{cases} \frac{x}{\sigma^2} e^{-\frac{1}{2}x^2/\sigma^2} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

EASILY INTEGRATED ANALYTICALLY. ARISES FROM SQUARE ROOT OF SUM OF SQUARES OF TWO INDEPENDENT $N(0,\sigma^2)$ RVS.

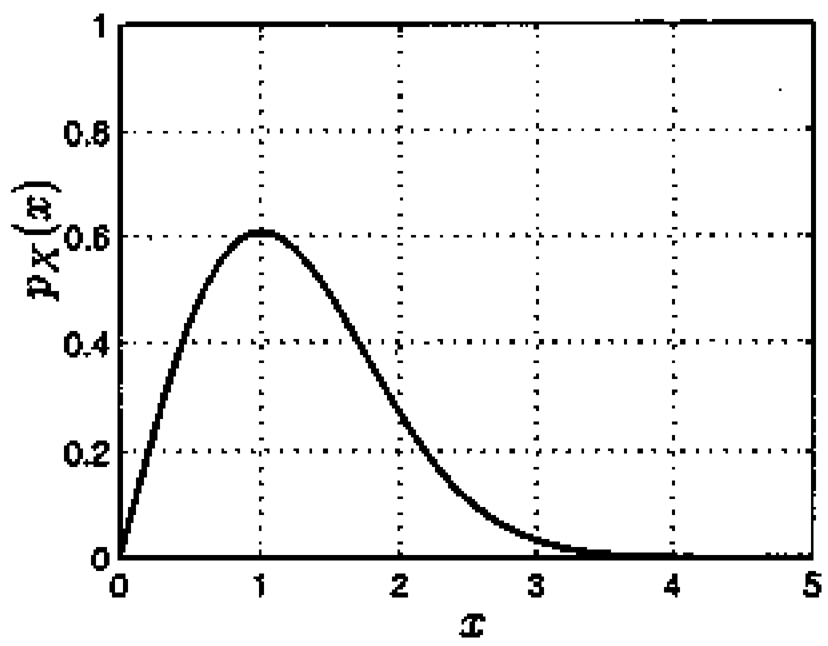


Figure 10.13: Rayleigh PDF with $\sigma^2 = 1$.

CUMULATIVE DISTRIBUTION FUNCTIONS (CDF)

USEFUL FOR FINDING $P[a \le X \le b]$ - "AVOIDS" INTEGRATION

DEFINED AS $F_X(x) = P[X \le x] \quad -\infty < x < \infty$

EVALUATE USING

$$F_X(x) = \int_{-\infty}^{x} p_X(t)\,dt$$

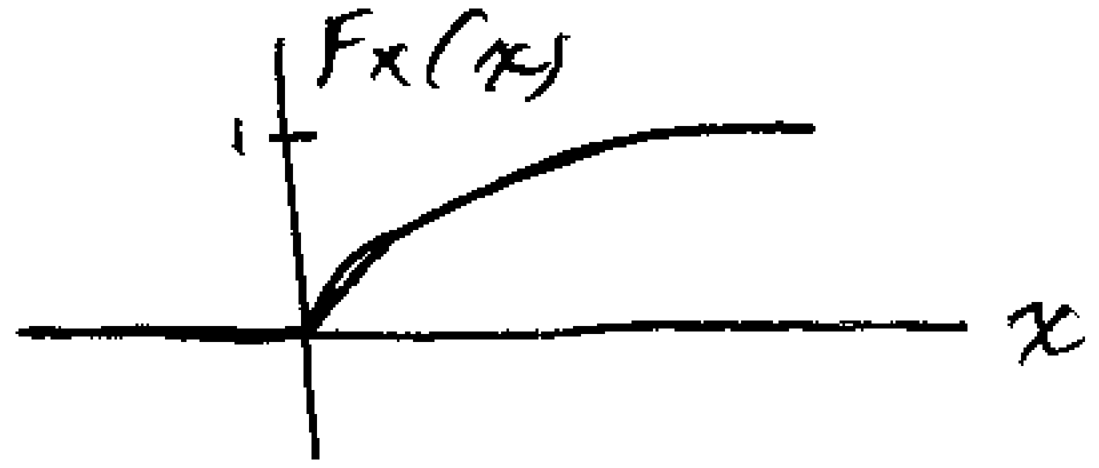
EXAMPLE : $X \sim EXP(\lambda)$

$$p_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & x < 0 \end{cases}$$

$$\Rightarrow \quad F_X(x) = P[X \le x] = 0 \qquad x < 0$$

$$= \int_0^x \lambda e^{-\lambda t}\,dt \qquad x \ge 0$$

$$= -e^{-\lambda t}\Big|_0^x = 1 - e^{-\lambda x}$$



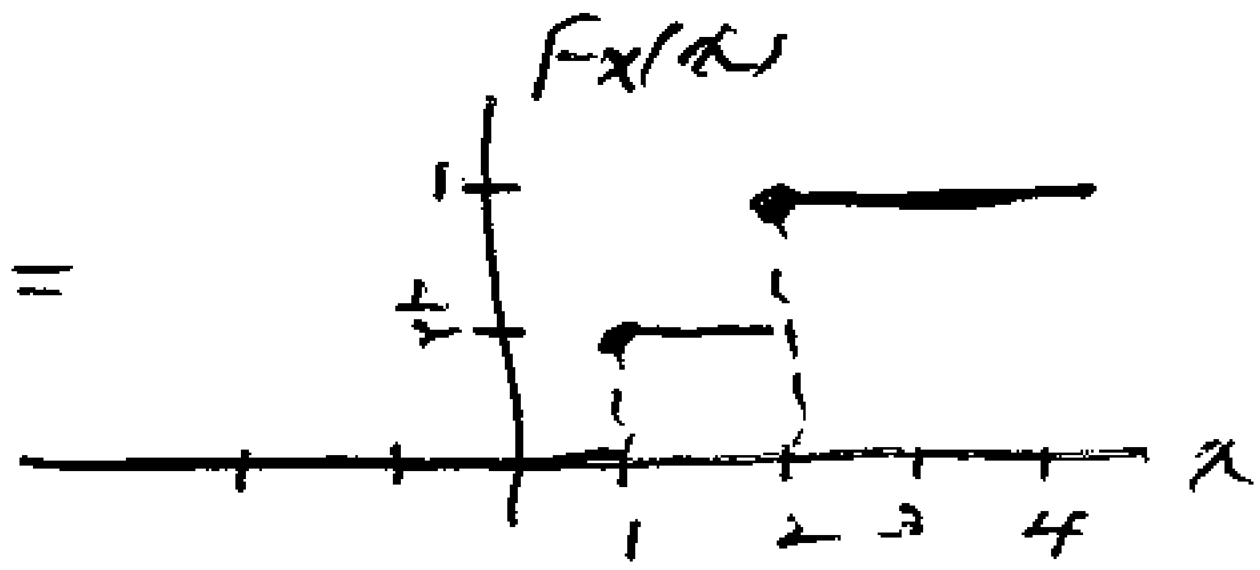
NOTE: $F_X(x)$ CONTINUOUS AT $x=0$ BUT NOT $p_X(x)$.

RANDOM VARIABLES ARE CALLED <u>CONTINUOUS</u> IF CDF IS CONTINUOUS

<u>EXAMPLE</u>: $p_X[k] = \frac{1}{2}$ $k=1$ (SECT 5.8)
$\frac{1}{2}$ $k=2$

DISCRETE R.V.

$F_X(x) = P[X \le x] =$



$$\begin{aligned} F_X(1) &= P[X \le 1] \\ &= P[X < 1 \text{ OR } X = 1] \\ &= P[X<1] + P[X=1] \\ &= 0 + \tfrac{1}{2} = \tfrac{1}{2} \end{aligned}$$

CALLED RIGHT-CONTINUOUS BUT OVERALL FOR DISCRETE R.V. CDF IS <u>DISCONTINUOUS</u>.

<u>EXAMPLE</u>: GAUSSIAN

$X \sim N(0,1)$

$$\Rightarrow p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \qquad -\infty < x < \infty$$

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

$$= \Phi(x)$$ SPECIAL SYMBOL

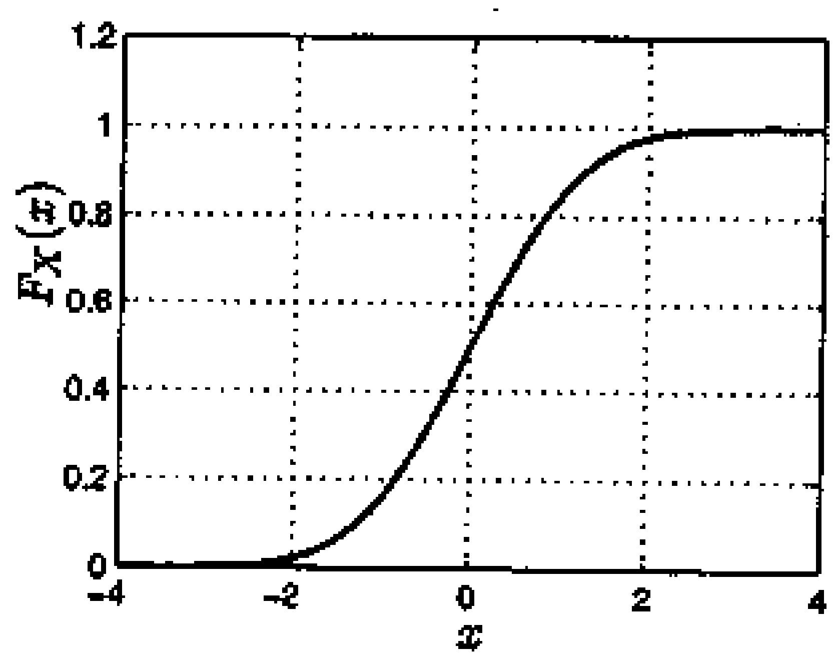
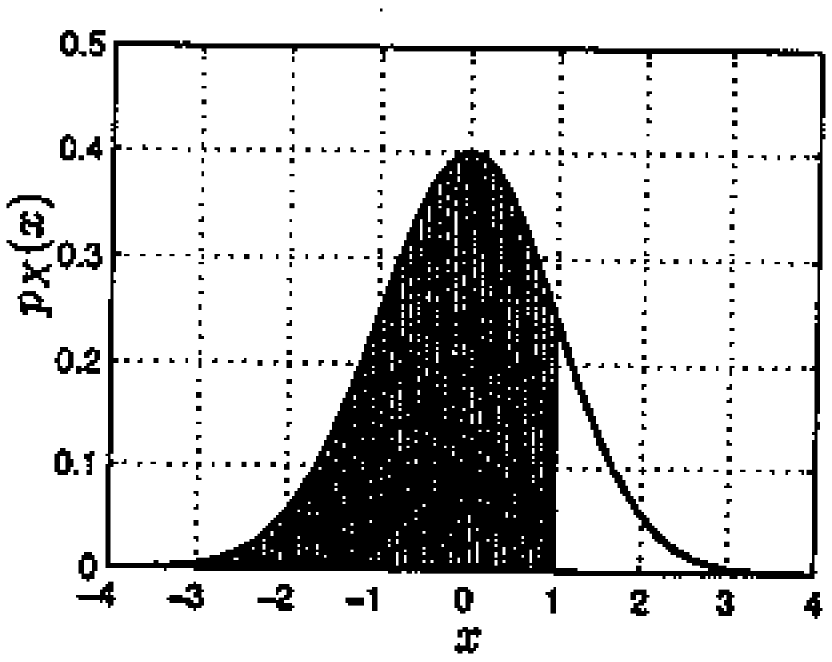


Figure 10.16: CDF for standard normal or Gaussian random variable.

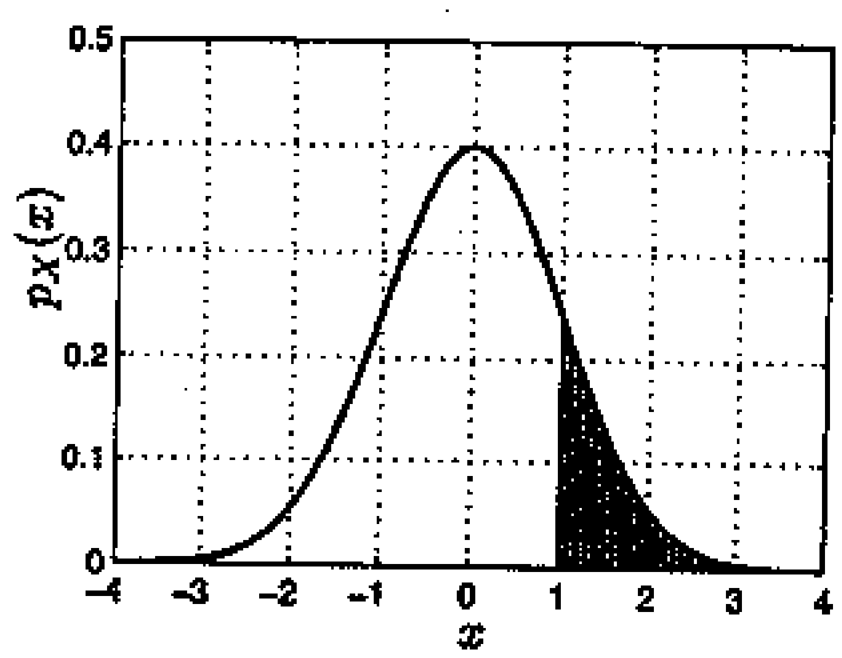
NOTE:
$F_X(-\infty) = 0$
$F_X(\infty) = 1$
WHY?
$F_X(0) = \frac{1}{2}$
WHY?

CAN'T BE FOUND ANALYTICALLY.

MORE CONVENIENT TO USE "RIGHT-TAIL" PROBABILITY FUNCTION $Q(x) = 1 - \Phi(x)$
$= P[X > x] = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$



(a) Shaded area = $\Phi(1)$



(b) Shaded area = $Q(1)$

Figure 10.17: Definitions of $\Phi(x)$ and $Q(x)$ functions.

SEE Q.m IN APPENDIX 10B (ALSO Qinv.m)

NOTE! $Q(-\infty) = 1$, $Q(\infty) = 0$
$Q(-x) = 1 - Q(x)$

EXAMPLE : PSK COMMUNICATION SYSTEM

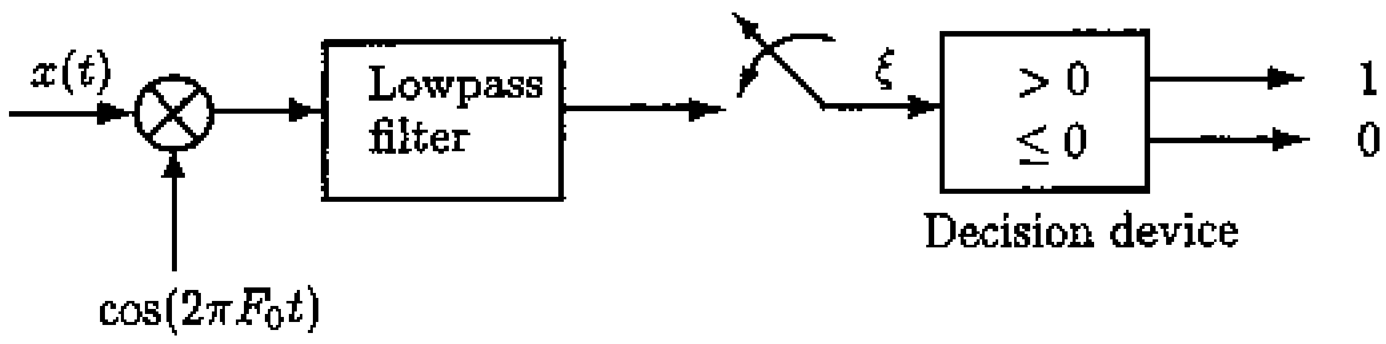


Figure 2.14: Receiver for a PSK digital communication system.

TRANSMIT $s_0(t) = A\cos(2\pi F_0 t + \pi)$
$= -A\cos(2\pi F_0 t)$ FOR "0"
$s_1(t) = A\cos(2\pi F_0 t)$ FOR "1"

CALLED BINARY PHASE SHIFT KEYING (PSK)

RECEIVE $x(t) = s_i(t) + w(t)$
↑ NOISE (MODELED AS RANDOM)

AT OUTPUT OF MULTIPLIER

$= A\cos(2\pi F_0 t + \pi)\cos(2\pi F_0 t)$ "0"
$= -A\left(\frac{1}{2} + \frac{1}{2}\cos(4\pi F_0 t)\right)$

$= A\cos(2\pi F_0 t)\cos(2\pi F_0 t)$
$= A\left(\frac{1}{2} + \frac{1}{2}\cos(4\pi F_0 t)\right)$ "1"

LOWPASS FILTER FILTERS OUT $\cos 4\pi F_0 t$
$\Rightarrow$ $\xi = -\frac{A}{2} + w$ "0"
$\frac{A}{2} + w$ "1"
DUE TO NOISE

ASSUME 1 IS SENT. WE HAVE AN ERROR IF $z \leq 0$. WHAT IS PROBABILITY OF THIS ERROR?

$$P_e = P[z \leq 0 \mid 1 \text{ SENT}]$$
$$= P[A/2 + W \leq 0] \qquad \text{ASSUME } W \sim N(0,1)$$

$$\Rightarrow \quad P_e = P[W \leq -A/2] = 1 - P[W > -A/2]$$
$$= 1 - Q(-A/2) = Q(A/2)$$

SEE FIG. 2.15 FOR PLOT
FOR $P_e \leq 0.1$ NEED $A > 2.6$

NOTE THAT $Q(x)$ FOR $x > 3$ IS WELL APPROXIMATED BY $Q(x) \approx \frac{1}{\sqrt{2\pi}\,x} e^{-\frac{1}{2}x^2}$
SEE FIG. 10.19

IN GENERAL, IF $X \sim N(\mu, \sigma^2)$, THEN
$$P[X > x] = Q\left(\frac{x-\mu}{\sqrt{\sigma^2}}\right)$$

(FOR STANDARD NORMAL $\mu = 0$, $\sigma^2 = 1$
$\Rightarrow \quad P[X > x] = Q(x)$).

RECALL MOTIVATION FOR CDF TO AVOID INTEGRATION. ONCE CDF IS FOUND NO FURTHER INTEGRATION REQUIRED.

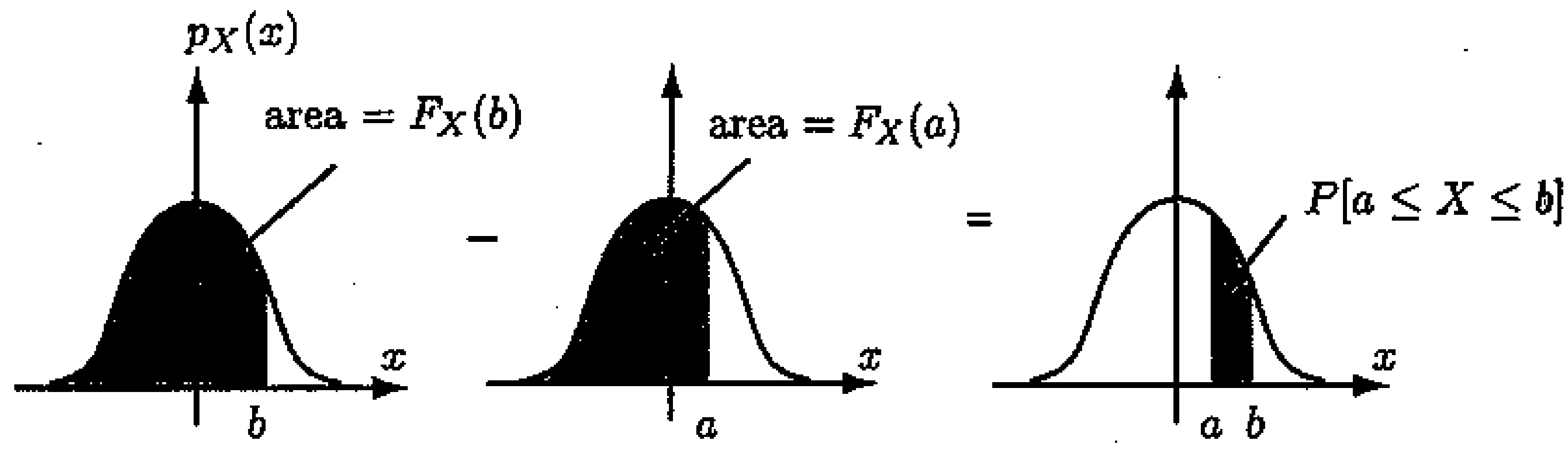


Figure 10.20: Illustration of use of CDF to find probability of interval.

$$P[a \le X \le b] = P[a < X \le b] \qquad \text{WHY?}$$
$$= F_X(b) - F_X(a)$$

EXAMPLE: $X \sim \text{EXP}(\lambda)$

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \ge 0 \\ 0 & x < 0 \end{cases}$$

$$\begin{aligned} P[a \le X \le b] &= F_X(b) - F_X(a) \qquad a > 0,\ b > 0 \\ &= (1 - e^{-\lambda b}) - (1 - e^{-\lambda a}) \\ &= e^{-\lambda a} - e^{-\lambda b} \\ &\stackrel{?}{=} \int_a^b p_X(x)\,dx \end{aligned}$$

CAN WE RECOVER PDF FROM CDF?

$$F_X(x) = \int_{-\infty}^{x} p_X(t)\,dt$$

$$\Rightarrow \quad p_X(x) = \frac{dF_X(x)}{dx} \qquad \text{FUNDAMENTAL THEOREM OF CALCULUS}$$

SEE ARGUMENT IN BOOK.

FOR ABOVE EXAMPLE, $F_X(x)$ IS DIFFERENTIABLE EVERYWHERE BUT AT $x = 0$ WHY? (FIG. 10.15)

$$p_X(x) = \frac{d(0)}{dx} \qquad x < 0$$
$$= 0$$
$$= \frac{d\,(1 - e^{-\lambda x})}{dx} \qquad x > 0$$
$$= \lambda e^{-\lambda x}$$

TRANSFORMATIONS

IMPORTANT TO BE ABLE TO FIND PDF OF $Y = g(X)$, IF PDF OF X IS KNOWN, FOR EXAMPLE, $Y = 2X$, $X \sim U(1, 2)$.

EXAMPLE: DISCRETE R.V.
$S_X = \{1, 2\}$ SAMPLE SPACE

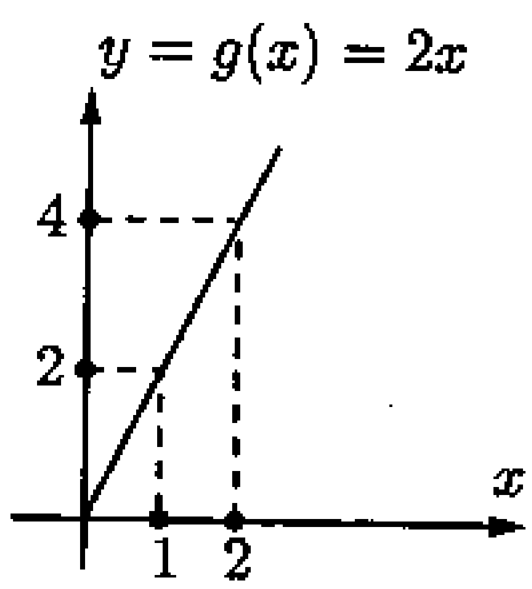


Figure 10.21: Transformation of a discrete random variable.

$S_Y = \{2, 4\}$
$p_Y[2] = p_X[1]$
$p_Y[4] = p_X[2]$
BUT $1 = g^{-1}(2)$
$2 = g^{-1}(4)$

$$\Rightarrow \quad p_Y[2] = p_X[g^{-1}(2)]$$
$$p_Y[4] = p_X[g^{-1}(4)]$$

IN GENERAL, $p_Y(y_i) = p_X[g^{-1}(y_i)]$
(ASSUMES ONE-TO-ONE TRANSFORMATION).

HOW DO WE EXTEND THIS TO CONT. RVS?

EXAMPLE: $Y = 2X \quad X \sim U(1,2)$

$\Rightarrow S_X = \{x : 1 < x < 2\}$

$S_Y = \{y : 2 < y < 4\}$

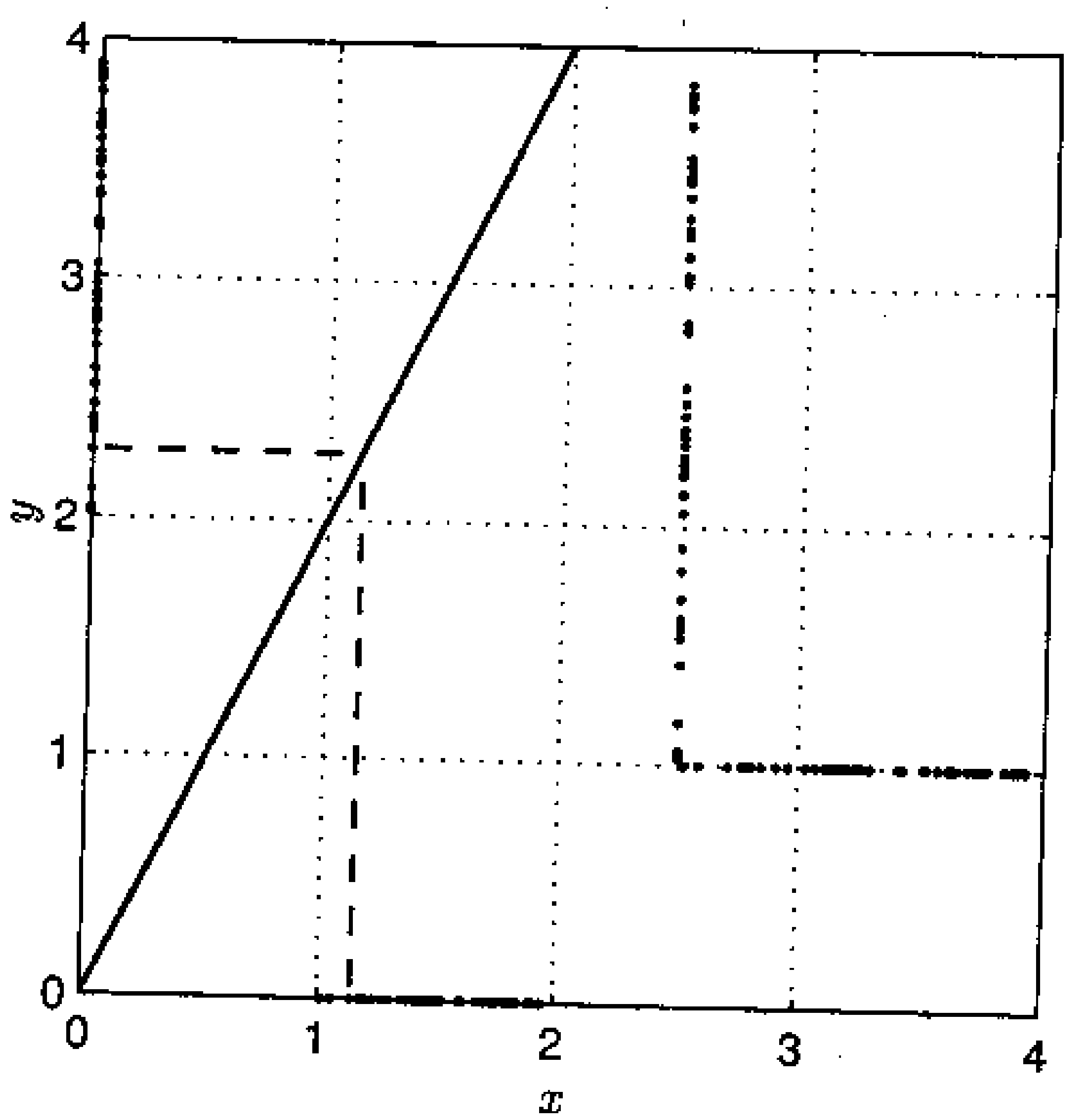
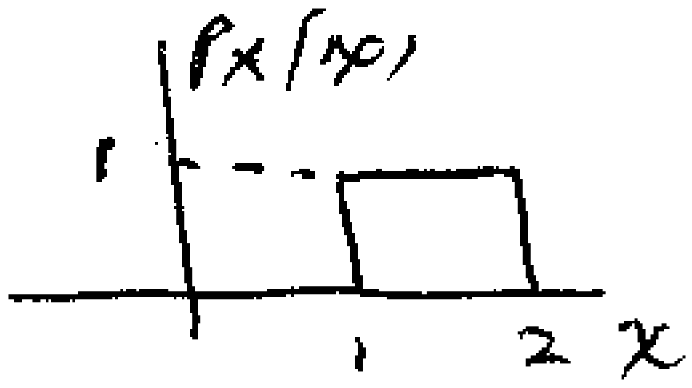
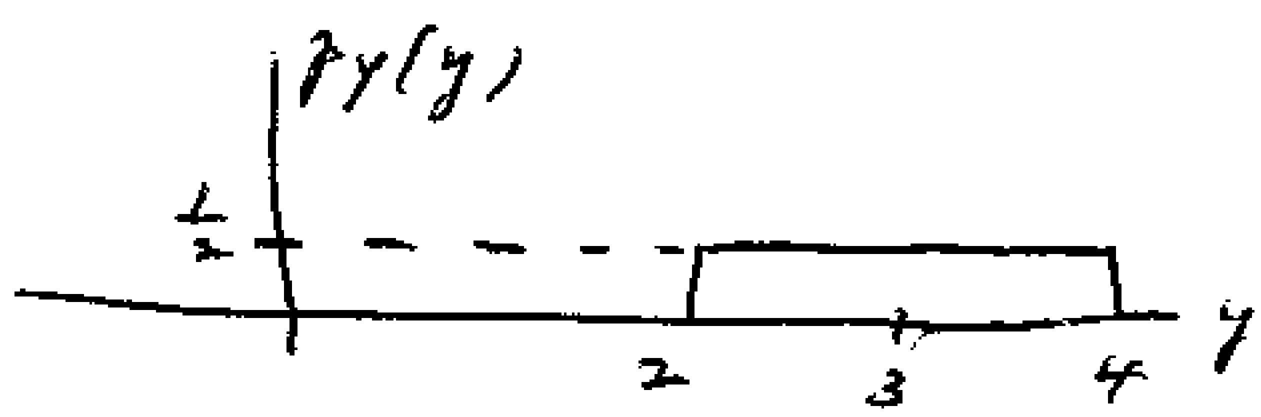


Figure 10.22: Computer generated realizations of X and $Y = 2X$ for $X \sim \mathcal{U}(1,2)$. A 50% expanded version of the realizations is shown to the right.

APPEARS AS IF $Y \sim U(2,4)$ OR





$$\Rightarrow \quad p_Y(y) = p_X(g^{-1}(y))\, \tfrac{1}{2}$$

SCALING FACTOR NOW (NOT NEEDED FOR DISCRETE R.V.)

ALSO, NOTE $\dfrac{dx}{dy} = \dfrac{d(\frac{1}{2}y)}{dy} = \frac{1}{2}$

SINCE $x = g^{-1}(y) = \frac{1}{2}y$

$$\Rightarrow \quad p_Y(y) = p_X(g^{-1}(y)) \left| \frac{d\, g^{-1}(y)}{dy} \right|$$

NEED FOR $p_Y > 0$.

SEE ARGUMENT IN BOOK.

<u>EXAMPLE</u>: $y = ax + b$

$S_X = \{x : -\infty < x < \infty\}$

$\Rightarrow \; S_Y = \{y : -\infty < y < \infty\} \quad a \neq 0$

(SHOULD ALWAYS FIND S_Y FIRST)

$y = g(x) = ax + b$

TO FIND g^{-1} SOLVE FOR x

$$x = \frac{y-b}{a} = g^{-1}(y)$$

$$\Rightarrow \quad p_Y(y) = p_X(g^{-1}(y)) \left| \frac{d\, g^{-1}(y)}{dy} \right|$$

$$= p_X\left(\frac{y-b}{a}\right) \left| \frac{1}{a} \right|$$

APPLICATION: IF $Y = \underbrace{\sqrt{\sigma^2}}_{a} X + \underbrace{\mu}_{b}$ $\quad X \sim N(0,1)$

$$\Rightarrow \quad p_Y(y) = p_X\left(\frac{y-\mu}{\sigma}\right)\left|\frac{1}{\sigma}\right| \qquad \sigma = \sqrt{\sigma^2} > 0$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} \frac{1}{\sigma}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$$

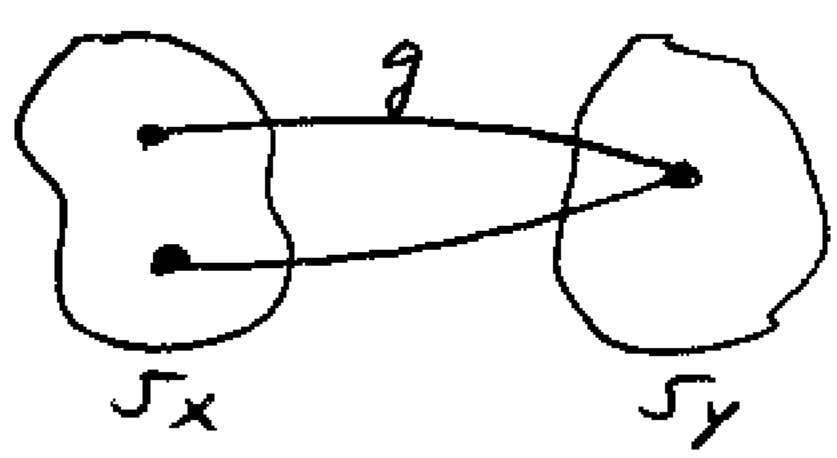
$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y-\mu)^2}$$

$Y \sim N(\mu, \sigma^2)$

IN MATLAB y = SQRT(SIGMA2) * randn(1,1) + mu

ALSO, LINEAR (AFFINE) TRANSFORMATION OF GAUSSIAN R.V. YIELDS ANOTHER GAUSSIAN R.V.

NOW WHAT HAPPENS IF TRANSFORMATION IS NOT ONE-TO-ONE.



TWO-TO-ONE, $g(x) = x^2$

FOR DISCRETE RVS WE HAVE AS AN EXAMPLE $S_X = \{-2, -1, 0, 1, 2\}$ $\quad Y = X^2$

$$S_Y = \{4, 1, 0\} = \{0, 1, 4\}$$

$p_Y(4) = p_X(-2) + p_X(2)$
$p_Y(1) = p_X(-1) + p_X(1)$
$p_Y(0) = p_X(0)$

OR LET $g_1^{-1}(4) = -2$
$g_2^{-1}(4) = 2$

$\Rightarrow p_Y(4) = p_X[g_1^{-1}(4)] + p_X[g_2^{-1}(4)]$
$p_Y(y) = p_X[g_1^{-1}(y)] + p_X[g_2^{-1}(y)]$

FOR CONT. RV. $Y = g(X)$ FOR WHICH THERE ARE TWO SOLUTIONS FOR X

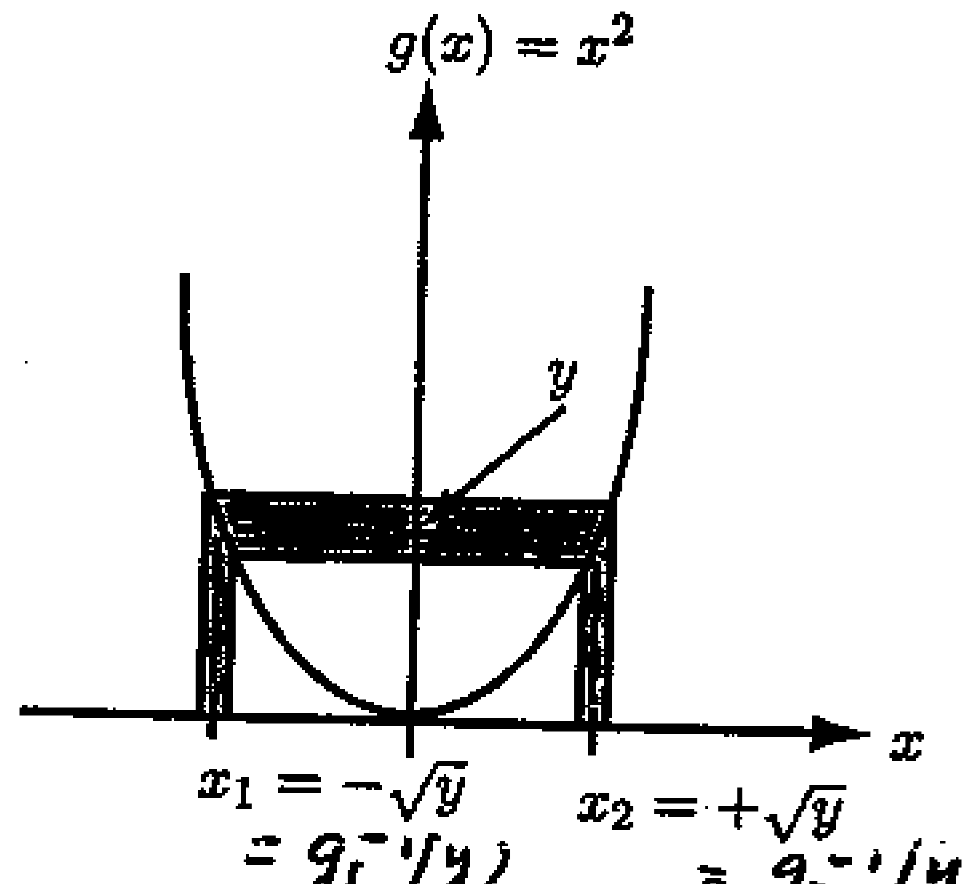


Figure 10.23: Solutions for x in $y = g(x) = x^2$.

$$p_Y(y) = p_X(g_1^{-1}(y)) \left| \frac{d g_1^{-1}(y)}{dy} \right| + p_X(g_2^{-1}(y)) \left| \frac{d g_2^{-1}(y)}{dy} \right|$$

EXAMPLE: $Y = X^2 \quad X \sim N(0, 1)$
$-\infty < X < \infty \Rightarrow Y \geq 0$
$g_1^{-1}(y) = -\sqrt{y} \quad g_2^{-1}(y) = +\sqrt{y}$

$$p_Y(y) = \begin{cases} p_X(-\sqrt{y}) \left|\frac{d(-\sqrt{y})}{dy}\right| + p_X(\sqrt{y}) \left|\frac{d(\sqrt{y})}{dy}\right| & y \geq 0 \\ 0 & y < 0 \end{cases}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-\sqrt{y})^2} \left|-\frac{1}{2\sqrt{y}}\right| \qquad y \geq 0$$

$$+ \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\sqrt{y})^2} \left|\frac{1}{2\sqrt{y}}\right|$$

$$= \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{-\frac{1}{2}y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

WHAT WOULD HAPPEN IF WE CONCLUDED

$$p_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-\frac{1}{2}y} \qquad -\infty < y < \infty \quad ?$$

<u>CDF APPROACH</u>

CAN SOMETIMES USE FOLLOWING STRATEGY:

1) FIND $F_Y(y)$
2) $p_Y(y) = dF_Y(y)/dy$

<u>EXAMPLE</u>: PREVIOUS ONE

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y)$$

$$= P(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$p_Y(y) = \frac{d}{dy}\left[F_X(\sqrt{y}) - F_X(-\sqrt{y})\right]$$

$$= \frac{dF_X(z_1)}{dz_1}\frac{dz_1}{dy} - \frac{dF_X(z_2)}{dz_2}\frac{dz_2}{dy} \qquad \begin{matrix} z_1 = \sqrt{y} \\ z_2 = -\sqrt{y} \end{matrix}$$

$$= p_X(z_1)\frac{dz_1}{dy} - p_X(z_2)\frac{dz_2}{dy}$$

$$= p_X(\sqrt{y})\frac{1}{2\sqrt{y}} - p_X(-\sqrt{y})\left(-\frac{1}{2\sqrt{y}}\right)$$

$$= p_X(\sqrt{y})\frac{1}{2\sqrt{y}} + p_X(-\sqrt{y})\frac{1}{2\sqrt{y}}$$

PROCEED AS BEFORE!

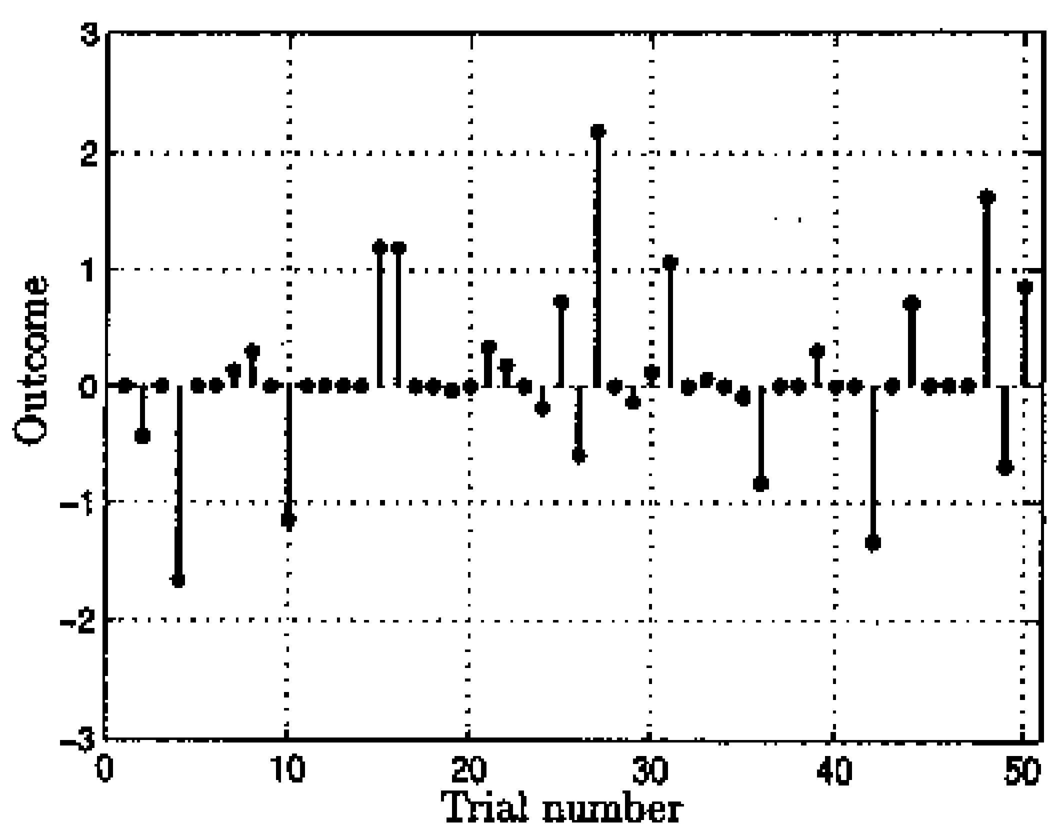
MIXED RANDOM VARIABLES

CAN HAVE RVS WITH CDFS THAT ARE CONTINUOUS BUT WITH ISOLATED JUMPS (DISCONTINUITIES).

EXAMPLE: TOSS COIN

IF HEADS $X \sim N(0,1)$

IF TAILS $X = 0$



FAIR COIN??

Figure 10.25: Sequence of outcomes for mixed random variable – $X = 0$ with nonzero probability.

CDF: $F_X(x) = P[X \le x]$

$= P[X \le x \mid \text{HEADS}]\, P[\text{HEADS}]$

$+ P[X \le x \mid \text{TAILS}]\, P[\text{TAILS}]$

ASSUME FAIR COIN

$F_X(x) = P[X \le x \mid X \sim N(0,1)]\, \frac{1}{2}$

$+ P[X \le x \mid X = 0]\, \frac{1}{2}$

$$= \Phi(x)\left(\tfrac{1}{2}\right) + \begin{cases} 0\left(\tfrac{1}{2}\right) & x < 0 \\ 1\left(\tfrac{1}{2}\right) & x \ge 0 \end{cases}$$

$$= \tfrac{1}{2}\Phi(x) + \tfrac{1}{2}u(x)$$

↑ UNIT STEP

$$= \begin{cases} 1 & x \ge 0 \\ 0 & x < 0 \end{cases}$$

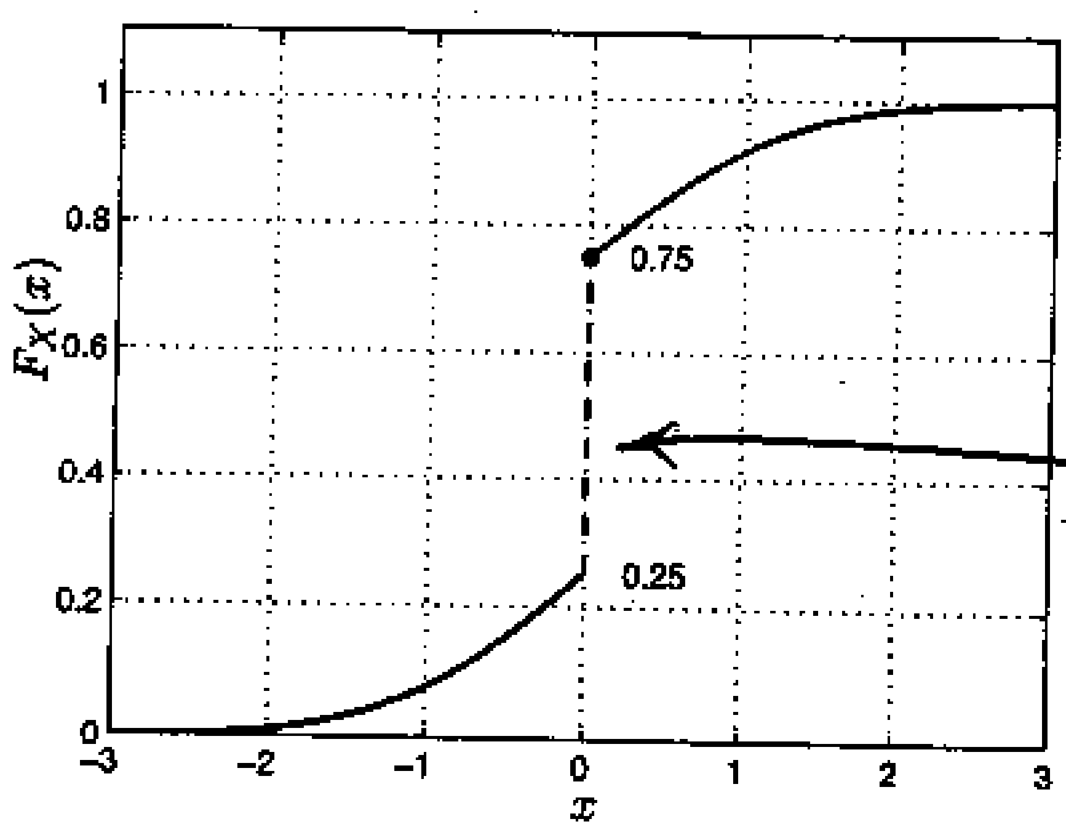


Figure 10.26: CDF for mixed random variable.

NOTE JUMP DUE TO $X = 0$ WITH NONZERO PROBABILITY. IN FACT

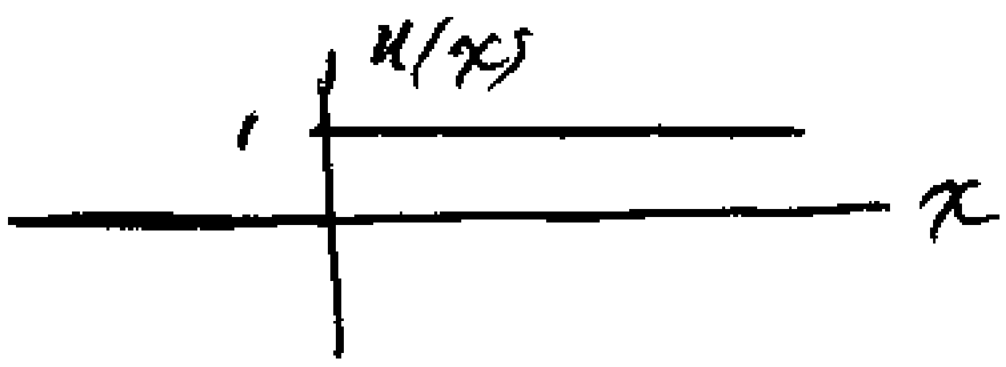
$P[X = 0] = F_X(0^+) - F_X(0^-)$

IF CONT $F_X(x)$, MUST HAVE $P[X = 0] = 0$

CALLED A <u>MIXED R.V.</u> . WHAT IS PDF?

$$p_X(x) = \frac{d}{dx}F_X(x) = \frac{d}{dx}\left[\tfrac{1}{2}\Phi(x) + \tfrac{1}{2}u(x)\right]$$

$$= \frac{1}{2} p_{N(0,1)}(x) + \frac{1}{2} \underbrace{\frac{du(x)}{dx}}_{= \,?}$$



$\frac{du(x)}{dx}$ <u>DEFINED</u> AS $\delta(x)$

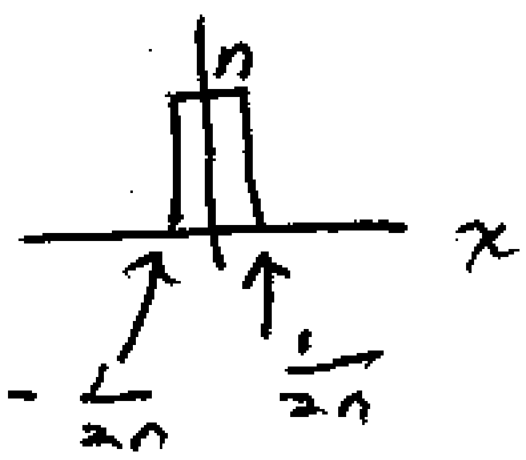
DIRAC IMPULSE FUNCTION

RECALL THAT

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

$$\int_{-\epsilon}^{\epsilon} \delta(x)\,dx = 1$$

$$= \lim_{n \to \infty}$$



$$\Rightarrow \quad p_X(x) = \frac{1}{2} p_{N(0,1)}(x) + \frac{1}{2}\delta(x)$$

$$= \frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} + \frac{1}{2}\delta(x)$$

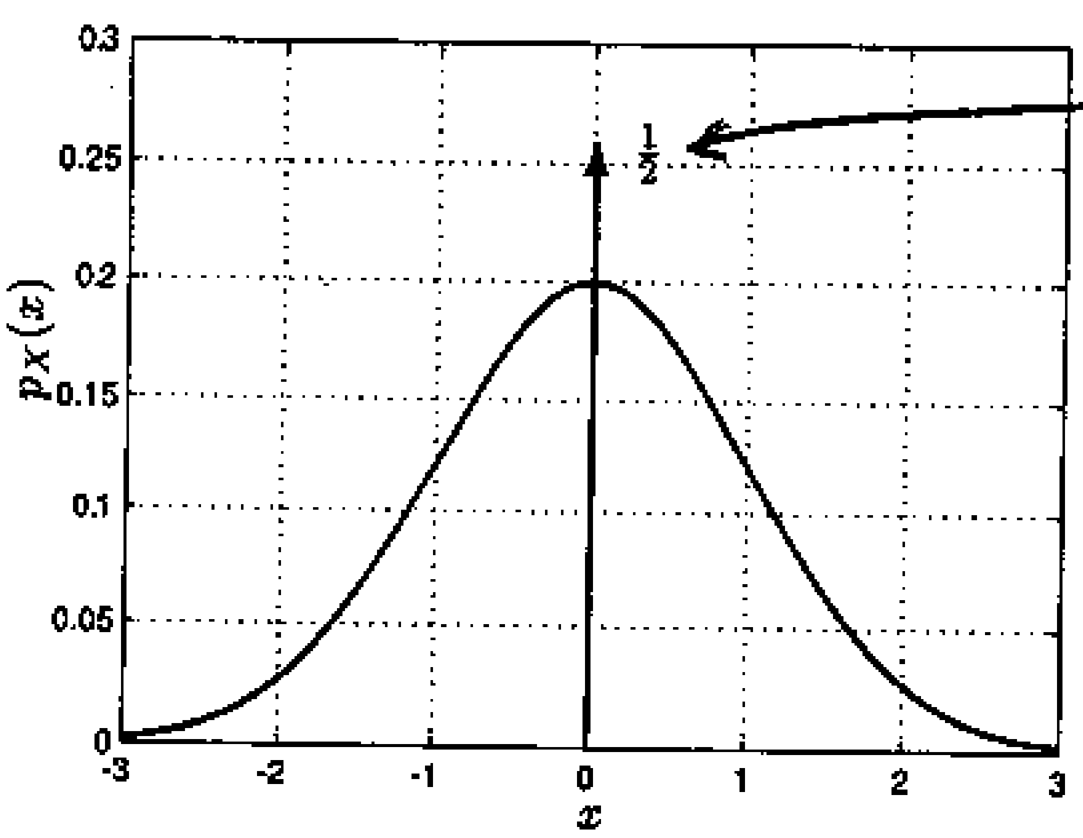


Figure 10.27: PDF for mixed random variable.

STRENGTH OF IMPULSE – <u>NOT</u> AMPLITUDE!

WHAT IS $P[X \leq 0]$?

" " $P[X < 0]$?

WHAT IS $\int_{-\infty}^{0} p_X(x)\,dx$?

WHEN IMPULSES ARE IN PDF, MUST BE CAREFUL.

$$P(X \le 0) = \int_{-\infty}^{0^+} p_X(x)\,dx = F_X(0^+)$$
$$P(X < 0) = \int_{-\infty}^{0^-} p_X(x)\,dx = F_X(0^-)$$

SEE FIG. 10.26.

IMPULSES FREQUENTLY ARISE DUE TO TRANSFORMATIONS, FOR EXAMPLE

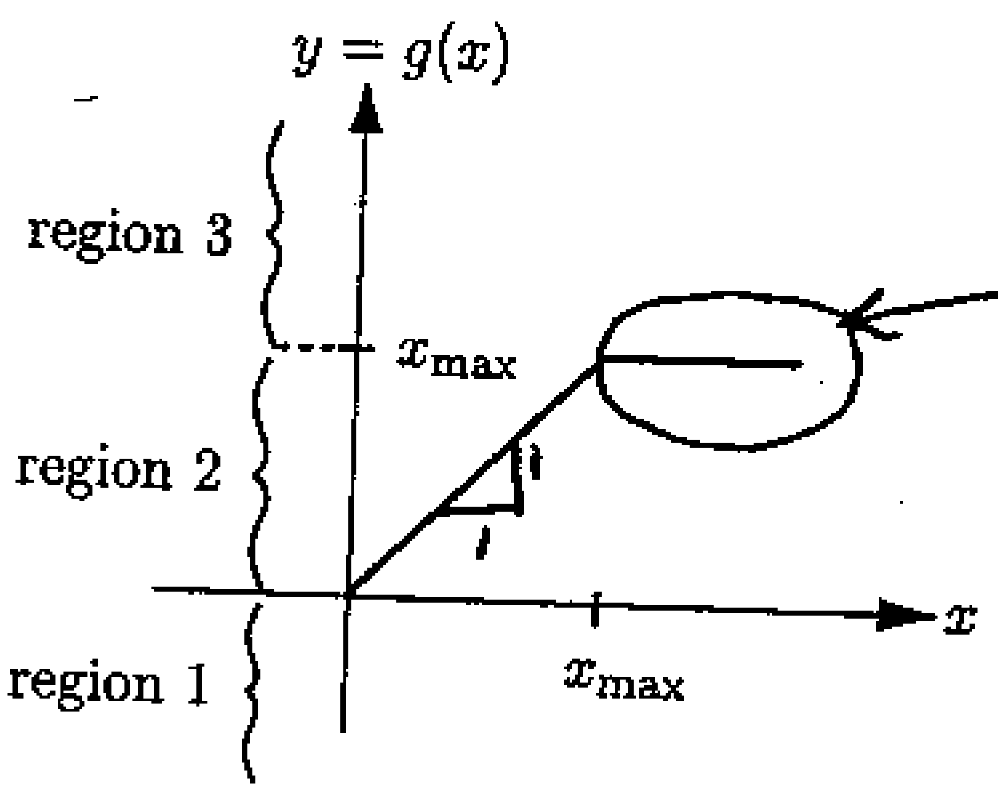


Figure 10.28: Amplitude limiter.

WILL YIELD IMPULSE IN $p_Y(y)$ AT $y = ?$

IN GENERAL, ALL PDFS CAN BE WRITTEN AS

$$p_X(x) = p_c(x) + \sum_{i=1}^{\infty} p_i\,\delta(x - x_i)$$

NO IMPULSES (CDF IS CONT.)

MUST HAVE $\int_{-\infty}^{\infty} p_c(x)\,dx + \sum_{i=1}^{\infty} p_i = 1$

COMPUTER SIMULATION

IN MATLAB rand $\Rightarrow$ $U(0,1)$
randn $\Rightarrow$ $N(0,1)$

HOW ABOUT GENERATING RVs WITH OTHER PDFs? $N(\mu, \sigma^2)$? $U(a,b)$?

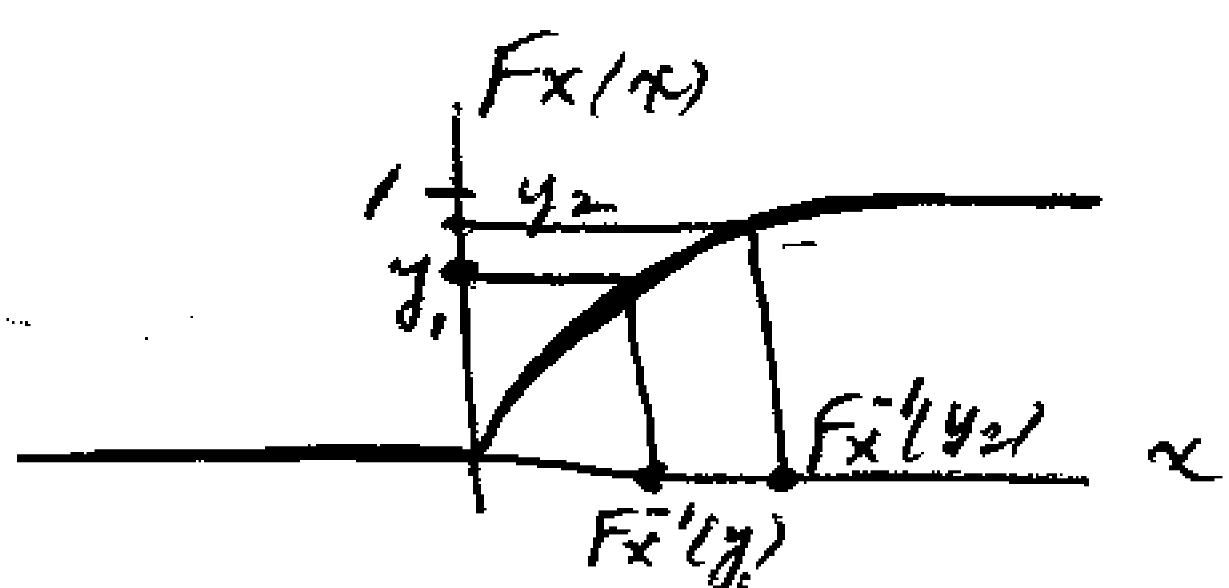
SOLUTION: TRANSFORM X VIA $Y = g(X)$ WHERE Y HAS DESIRED PDF. (SAY $X \sim U(0,1)$ IF WE USE rand)

LET $U = U(0,1)$ R.V. WE WISH TO FIND $g(\ )$ SO THAT $X = g(U)$ HAS DESIRED PDF. THE NECESSARY $g(\ )$ IS $F_X^{-1}(\ )$ SO THAT $X = F_X^{-1}(U)$.

$F_X^{-1}(\ )$ IS INVERSE CDF FUNCTION.

EXAMPLE: $X \sim EXP(\lambda)$

RECALL $F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$



NOTE THAT INPUT TO $F_X^{-1}(\ )$ IS $(0,1)$, THE POSSIBLE VALUES OF U. HENCE, WE NEED ONLY FIND $F_X^{-1}(y)$ FOR $0 < y < 1$. FOR THESE y'S, $x > 0$ AND
$y = F_X(x) = 1 - e^{-\lambda x}$

$$y = 1 - e^{-\lambda x}$$

$$1 - y = e^{-\lambda x} \Rightarrow x = -\frac{1}{\lambda} \ln(1-y)$$
$$F_X^{-1}(y) = -\frac{1}{\lambda} \ln(1-y)$$

HENCE $\quad X = F_X^{-1}(U)$
$$= -\frac{1}{\lambda} \ln(\underset{\uparrow\, U(0,1)}{1-U}) \sim EXP(\lambda).$$

ALSO, NOTE THAT IF $X = F_X^{-1}(U)$, THEN $U = F_X(X) \sim U(0,1)$. THIS IS CALLED THE <u>PROBABILITY INTEGRAL TRANSFORMATION</u> AND HENCE $F_X^{-1}(U)$ IS THE <u>INVERSE PROBABILITY INTEGRAL TRANSFORMATION</u>.

SEE THEOREM 10.9.1 FOR PROOF OF $X = F_X^{-1}(U)$ RESULT.

EXAMPLE: GENERATE LAPLACIAN R.V. OUTCOME

$$p_X(x) = \frac{1}{\sqrt{2\sigma^2}} e^{-\sqrt{\frac{2}{\sigma^2}}|x|} \qquad -\infty < x < \infty$$

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\sigma^2}} e^{-\sqrt{2/\sigma^2}|t|} dt$$

SEE BOOK $= \frac{1}{2} e^{\sqrt{2/\sigma^2}\, x} \qquad x < 0$

$1 - \frac{1}{2} e^{-\sqrt{2/\sigma^2}\, x} \qquad x \geq 0$

LET $y = F_X(x)$ AND SOLVE FOR x

FOR $x < 0$ $\quad y = \frac{1}{2} e^{\sqrt{2/\sigma^2}\, x}$

$$\Rightarrow \quad x = \sqrt{\sigma^2/2}\, \ln(2y)$$

NOTE: IF $x < 0$, $y = \frac{1}{2} e^{\sqrt{2/\sigma^2}\, x} < \frac{1}{2}$
ALSO, $y > 0$

$$F_X^{-1}(y) = \sqrt{\sigma^2/2}\, \ln(2y) \qquad 0 < y < \tfrac{1}{2}$$

SIMILARLY $F_X^{-1}(y) = \sqrt{\sigma^2/2}\, \ln\left(\frac{1}{2(1-y)}\right)$
FOR $\frac{1}{2} \leq y < 1$

HENCE, $x = F_X^{-1}(u)$ WILL BE OUTCOME OF LAPLACIAN RV IF u IS OUTCOME OF $U(0,1)$ RV.