BME464: Medical Imaging, Supplemental: Fourier and Euler, Explained

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Leonard Euler (1707-1783)

- Swiss
- Friends with Bernoulli
- Fourier is a pretty big deal

Joseph Fourier(1768-1830)

- French (Jean-Baptiste)
- Friends with Napoleon
- Euler is a bigger deal

"I knew Leonard Euler. And you, Mr. Fourier, are no Leonard Euler"

The Biggies

- Fourier Transform
- $S(f) = \int_{-\infty}^{\infty} s(t) \cdot e^{-i2\pi ft} dt$, but also
- $S[k] = \frac{1}{P} \int_P s_P(t) \cdot e^{-i2\pi \frac{k}{P} t} dt$
- Discrete Fourier Transform
- $$S_{1/T}(f) \stackrel{\text{def}}{=} \underbrace{\sum_{k=-\infty}^{\infty} S\left(f - \frac{k}{T}\right) \equiv \overbrace{\sum_{n=-\infty}^{\infty} s[n] \cdot e^{-i2\pi fnT}}^{\text{Fourier series (DTFT)}}}_{\text{Poisson summation formula}} =$$
  $\mathcal{F}\left\{\sum_{n=-\infty}^{\infty} s[n]\ \delta(t - nT)\right\}$
- Fourier Series
- $s_P(t) = \sum_{k=-\infty}^{\infty} S[k] \cdot e^{i2\pi \frac{k}{P} t} \quad \stackrel{\mathcal{F}}{\Longleftrightarrow} \quad \sum_{k=-\infty}^{+\infty} S[k]\ \delta\left(f - \frac{k}{P}\right)$

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Euler

The Biggies

- $e^{i\varphi} = \cos\varphi + i\sin\varphi$ (Euler's Formula)
- $e^{i\pi} + 1 = 0$ (Euler's Identity, special case of above)
- Why?
- This means $e^{i\pi} = -1$, which we all know (right?)
- And so we begin

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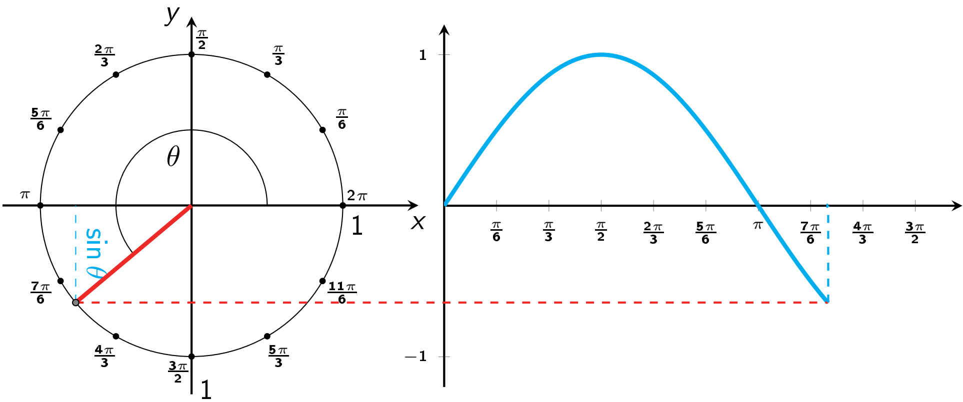
Unit Circle



y
x
1
1
θ
sin θ
π/2
π/3
π/6
2π/3
5π/6
π
2π
7π/6
11π/6
4π/3
3π/2
5π/3
1
−1
π/6
π/3
π/2
2π/3
5π/6
π
7π/6
4π/3
3π/2

z-transform

- $e^z = \lim_{N \to \infty} (1 + z/N)^N$
- (Yeah, that's Euler)
- Make the substitution, $z = i\pi$

To MATLAB

z-transform

```
clear all
for n = 1:100
euler(n) = (1+1i*pi./n).^n;
zplane(euler(n)),axis([-4 4 -4 4]),hold on,pause
end

clear all
for n = 1:100
euler(n) = (1+1i*pi./n).^n;
zplane(euler),axis([-2 2 -2 2]),pause
end
```

DTFT

```
%clear all
%create a time vector
t = −1:0.001:1;
%make a 10 hz sine wave
sig = sin(2*pi*10*t);
%look what happens when we vary n (which controls the frequenc
for n = 1:20
    ft = sig.*exp(−j*2*pi*n*t);
    plot3(t,real(ft),imag(ft)),pause(1)
    %this is an approximation to the Fourier transform − look
    summation = abs(sum(ft))
    if n == 10
        %you have to press any key to break out of the pause
        %this lets us look at this particular frequency
        pause
    end
end
```

FT Symmetries

$$x[n] = \operatorname{Re} x[n] + \operatorname{Im} x[n] \begin{cases} \operatorname{Re} x[n] = \frac{1}{2}\left(x[n] + \overline{x[n]}\right) \\ \operatorname{Im} x[n] = \frac{1}{2j}\left(x[n] - \overline{x[n]}\right) \end{cases} \tag{1}$$

$$\operatorname{Re} x[n] = \operatorname{Re} x_e[n] + \operatorname{Re} x_o[n] \begin{cases} \operatorname{Re} x_e[n] = \frac{1}{2}\left(\operatorname{Re} x[n] + \operatorname{Re} x[-n]\right) \\ \operatorname{Re} x_o[n] = \frac{1}{2}\left(\operatorname{Re} x[n] - \operatorname{Re} x[-n]\right) \end{cases} \tag{2}$$

$$\operatorname{Im} x[n] = \operatorname{Im} x_e[n] + \operatorname{Im} x_o[n] \begin{cases} \operatorname{Im} x_e[n] = \frac{1}{2}\left(\operatorname{Im} x[n] + \operatorname{Im} x[-n]\right) \\ \operatorname{Im} x_o[n] = \frac{1}{2}\left(\operatorname{Im} x[n] - \operatorname{Im} x[-n]\right) \end{cases} \tag{3}$$

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So ...

$$x[n] = \mathrm{Re}\, x_e[n] + \mathrm{Re}\, x_o[n] + \mathrm{Im}\, x_e[n] + \mathrm{Im}\, x_o[n]$$

$$\mathrm{Re}\, x_e[n] = \frac{1}{4}\left(x[n] + \overline{x[n]} + x[-n] + \overline{x[-n]}\right) \tag{4}$$

$$\mathrm{Re}\, x_o[n] = \frac{1}{4}\left(x[n] + \overline{x[n]} - x[-n] - \overline{x[-n]}\right) \tag{5}$$

$$\mathrm{Im}\, x_e[n] = \frac{1}{4j}\left(x[n] - \overline{x[n]} + x[-n] - \overline{x[-n]}\right) \tag{6}$$

$$\mathrm{Im}\, x_o[n] = \frac{1}{4j}\left(x[n] - \overline{x[n]} - x[-n] + \overline{x[-n]}\right) \tag{7}$$

Symmetry extends to the Fourier Transform

$$X(\omega) = \operatorname{Re} X_e(\omega) + \operatorname{Re} X_o(\omega) + j(\operatorname{Im} X_e(\omega) + \operatorname{Im} X_o(\omega))$$

$$\begin{aligned}
\operatorname{Re} X_e(\omega) &= \frac{1}{4}\left(X(\omega) + \overline{(X(\omega)} + X(-\omega) + \overline{X(-\omega)}\right) && (8)\\
\operatorname{Re} X_o(\omega) &= \frac{1}{4}\left(X(\omega) + \overline{X(\omega)} - X(-\omega) - \overline{X(-\omega)}\right) && (9)\\
\operatorname{Im} X_e(\omega) &= \frac{1}{4j}\left(X(\omega) - \overline{X(\omega)} + X(-\omega) - \overline{X(-\omega)}\right) && (10)\\
\operatorname{Im} X_o(\omega) &= \frac{1}{4j}\left(X(\omega) - \overline{X(\omega)} - X(-\omega) + \overline{X(-\omega)}\right) && (11)
\end{aligned}$$

Because ...

$$x[n] \rightarrow X(\omega) \tag{12}$$

$$\overline{x[n]} \rightarrow \overline{X(-\omega)} \tag{13}$$

$$x[-n] \rightarrow X(-\omega) \tag{14}$$

$$\overline{x[-n]} \rightarrow \overline{X(\omega)} \tag{15}$$

Which means ...

$$\mathrm{Re}\, x_e[n] \rightarrow \mathrm{Re}\, X_e(\omega) \tag{16}$$

$$\mathrm{Re}\, x_o[n] \rightarrow \mathrm{Im}\, X_o(\omega) \tag{17}$$

$$\mathrm{Im}\, x_e[n] \rightarrow \mathrm{Im}\, X_e(\omega) \tag{18}$$

$$\mathrm{Im}\, x_o[n] \rightarrow \mathrm{Re}\, X_o(\omega) \tag{19}$$

... and ultimately $e^{j\omega t} = \overline{e^{j\omega(-t)}}$

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