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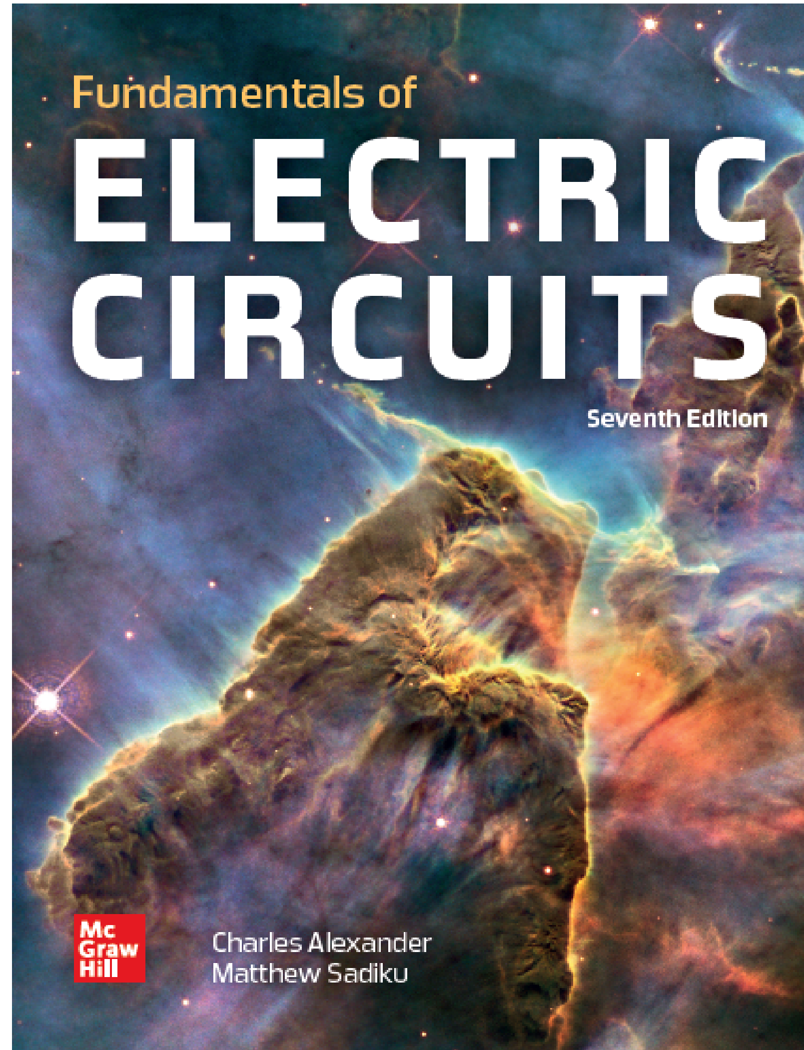
Fundamentals of Electric Circuits

Chapter 9



Overview

- This chapter will cover alternating current.
- A discussion of complex numbers is included prior to introducing phasors.
- Applications of phasors and frequency domain analysis for circuits including resistors, capacitors, and inductors will be covered.
- The concept of impedance and admittance is also introduced.

Alternating Current

- Alternating Current, or AC, is the dominant form of electrical power that is delivered to homes and industry.
- In the late 1800’s there was a battle between proponents of DC and AC.
- AC won out due to its efficiency for long distance transmission.
- AC is a sinusoidal current, meaning the current reverses at regular times and has alternating positive and negative values.

Sinusoids 1

- Sinusoids are interesting to us because there are a number of natural phenomenon that are sinusoidal in nature.
- It is also a very easy signal to generate and transmit.
- Also, through Fourier analysis, any practical periodic function can be made by adding sinusoids.
- Lastly, they are very easy to handle mathematically.

Sinusoids 2

- A sinusoidal forcing function produces both a transient and a steady state response.
- When the transient has died out, we say the circuit is in sinusoidal steady state.
- A sinusoidal voltage may be represented as:

$$v(t) = V_m \sin \omega t$$

- From the waveform shown below, one characteristic is clear: The function repeats itself every T seconds.
- This is called the period.

$$T = \frac{2\pi}{\omega}$$

Sinusoids 3

- The period is inversely related to another important characteristic, the frequency.

$$f = \frac{1}{T}$$

- The units of this is cycles per second, or Hertz (Hz).
- It is often useful to refer to frequency in angular terms:

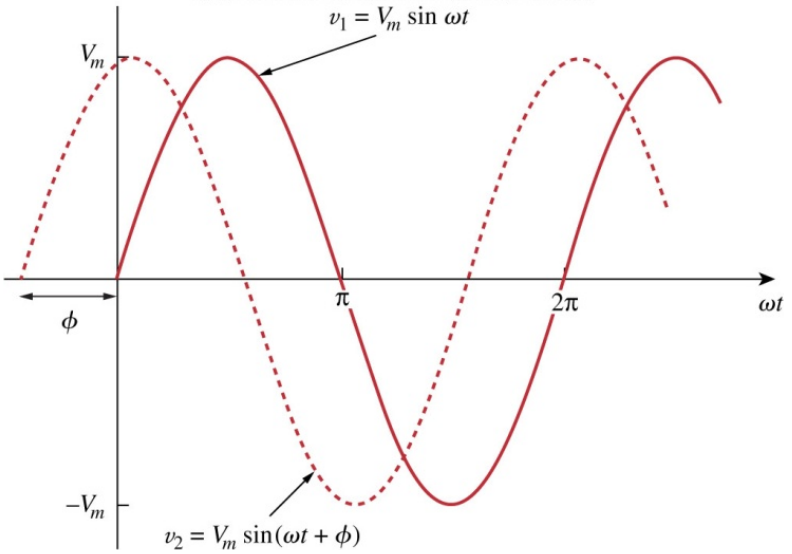
$$\omega = 2\pi f$$

- Here the angular frequency is in radians per second.

Sinusoids 4

- More generally, we need to account for relative timing of one wave versus another.
- This can be done by including a phase shift, ϕ:
- Consider the two sinusoids:

$$v_1(t) = V_m \sin \omega t \quad \text{and} \quad v_2(t) = V_m \sin(\omega t + \phi)$$



Sinusoids 5

- If two sinusoids are in phase, then this means that the reach their maximum and minimum at the same time.
- Sinusoids may be expressed as sine or cosine.
- The conversion between them is:

$$\sin\left(\omega t \pm 180^\circ\right) = -\sin \omega t$$

$$\cos\left(\omega t \pm 180^\circ\right) = -\cos \omega t$$

$$\sin\left(\omega t \pm 90^\circ\right) = \pm\cos \omega t$$

$$\cos\left(\omega t \pm 90^\circ\right) = \mp\sin \omega t$$

Complex Numbers 1

- A powerful method for representing sinusoids is the phasor.
- But in order to understand how they work, we need to cover some complex numbers first.
- A complex number *z* can be represented in rectangular form as:

$$z = x + jy$$

- It can also be written in polar or exponential form as:

$$z = r\angle\phi = re^{j\phi}$$

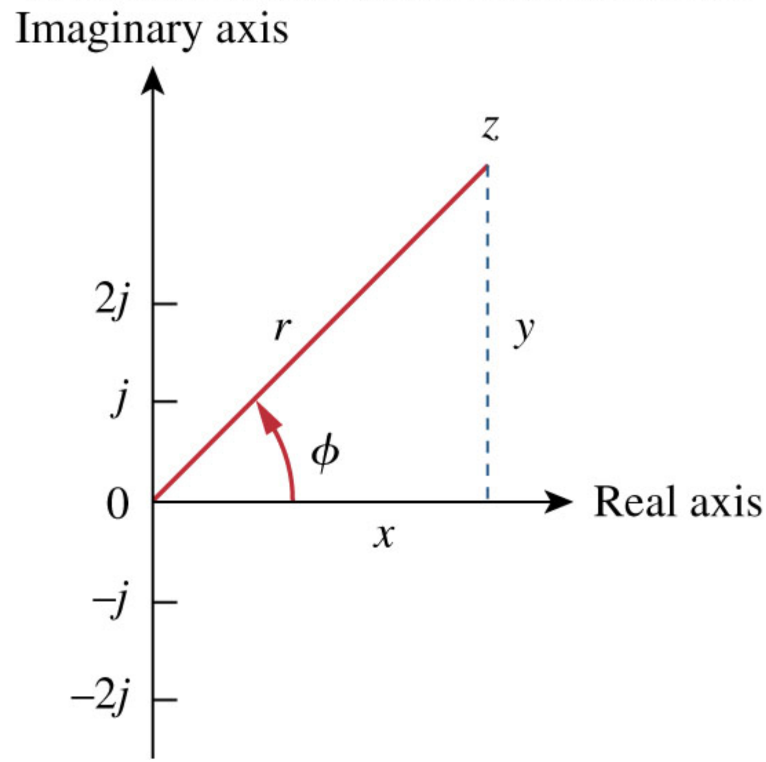
Complex Numbers 2

- The different forms can be interconverted.
- Starting with rectangular form, one can go to polar:

$$r = \sqrt{x^2 + y^2} \qquad \phi = \tan^{-1}\frac{y}{x}$$

- Likewise, from polar to rectangular form goes as follows:

$$x = r\cos\phi \qquad y = r\sin\phi$$



Complex Numbers 3

- The following mathematical operations are important.

Addition

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

Subtraction

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

Multiplication

$$z_1 z_2 = r_1 r_2 \angle (\phi_1 + \phi_2)$$

Division

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle (\phi_1 - \phi_2)$$

Reciprocal

$$\frac{1}{z} = \frac{1}{r} \angle (-\phi)$$

Square Root

$$\sqrt{z} = \sqrt{r} \angle (\phi / 2)$$

Complex Conjugate

$$z^* = x - jy = r \angle -\phi = r e^{-j\phi}$$

Phasors [1]

- The idea of a phasor representation is based on Euler's identity:

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$

- From this we can represent a sinusoid as the real component of a vector in the complex plane.
- The length of the vector is the amplitude of the sinusoid.
- The vector, V, in polar form, is at an angle ϕ with respect to the positive real axis.

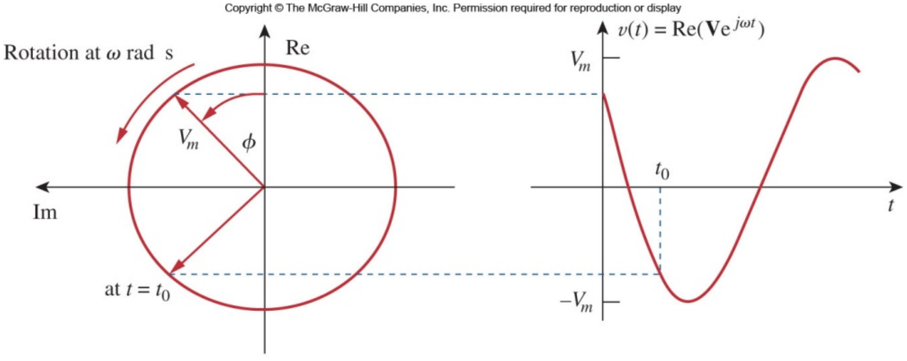
Phasors 2

- Phasors are typically represented at *t = 0*.
- As such, the transformation between time domain to phasor domain is:

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad V = V_m \angle \phi$$

(Time-domain representation) (Phasor-domain representation)

- They can be graphically represented as shown here.



(a) (b)

Sinusoid-Phasor Transformation 1

- Here is a handy table for transforming various time domain sinusoids into phasor domain:

TABLE 9.1

Sinusoid-phasor transformation.

Time domain representation	Phasor domain representation
$V_m \cos(\omega t + \phi)$	$V_m \angle \phi$
$V_m \sin(\omega t + \phi)$	$V_m \angle \phi - 90^\circ$
$I_m \cos(\omega t + \theta)$	$I_m \angle \theta$
$I_m \sin(\omega t + \theta)$	$I_m \angle \theta - 90^\circ$

Sinusoid-Phasor Transformation 2

- Note that the frequency of the phasor is not explicitly shown in the phasor diagram.
- For this reason phasor domain is also known as frequency domain.
- Applying a derivative to a phasor yields:

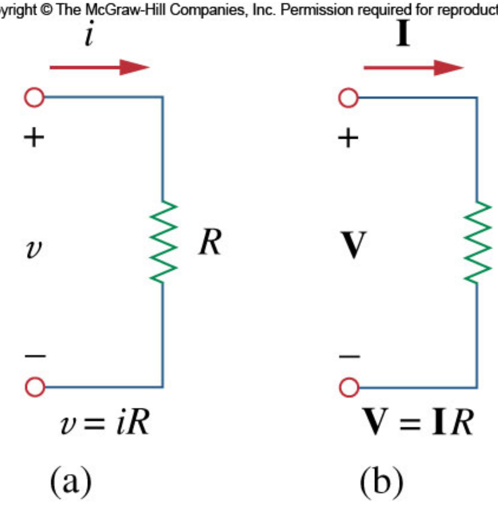
$$\underset{\text{(Time domain)}}{\frac{dv}{dt}} \quad \Leftrightarrow \quad \underset{\text{(Phasor domain)}}{j\omega V}$$

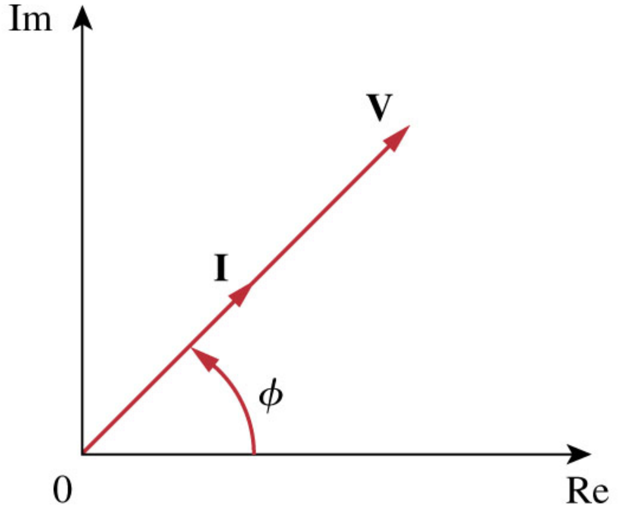
- Applying an integral to a phasor yields:

$$\underset{\text{(Time domain)}}{\int v\,dt} \quad \Leftrightarrow \quad \underset{\text{(Phasor domain)}}{\frac{V}{j\omega}}$$

Phasor Relationships for Resistors

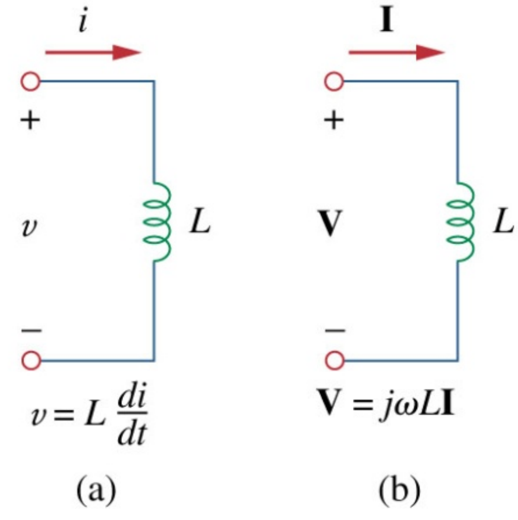
- Each circuit element has a relationship between its current and voltage.
- These can be mapped into phasor relationships very simply for resistors capacitors and inductor.
- For the resistor, the voltage and current are related via Ohm's law.
- As such, the voltage and current are in phase with each other.

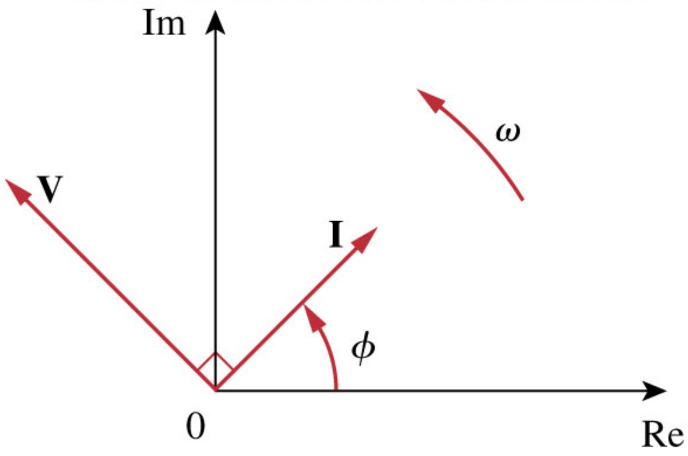




Phasor Relationships for Inductors

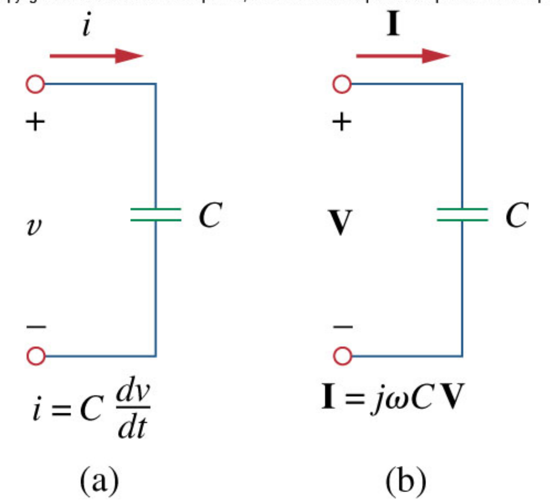
- Inductors on the other hand have a phase shift between the voltage and current.
- In this case, the voltage leads the current by 90°.
- Or one says the current lags the voltage, which is the standard convention.
- This is represented on the phasor diagram by a positive phase angle between the voltage and current.

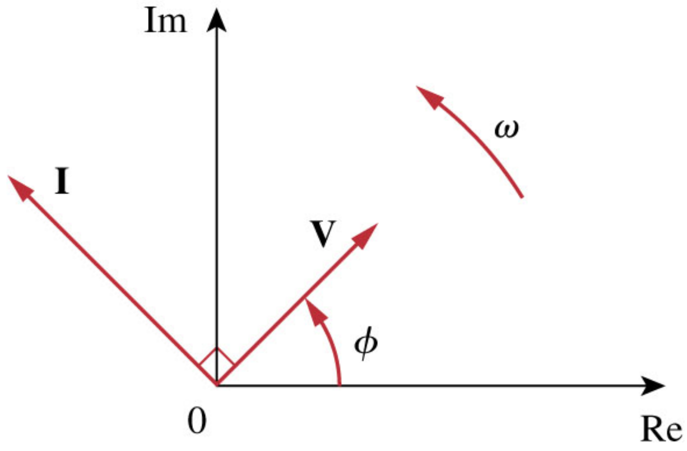




Phasor Relationships for Capacitors

- Capacitors have the opposite phase relationship as compared to inductors.
- In their case, the current leads the voltage.
- In a phasor diagram, this corresponds to a negative phase angle between the voltage and current.





Voltage current relationships

TABLE 9.2

Summary of voltage-current relationships.

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\dfrac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\dfrac{dv}{dt}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$

Impedance and Admittance [1]

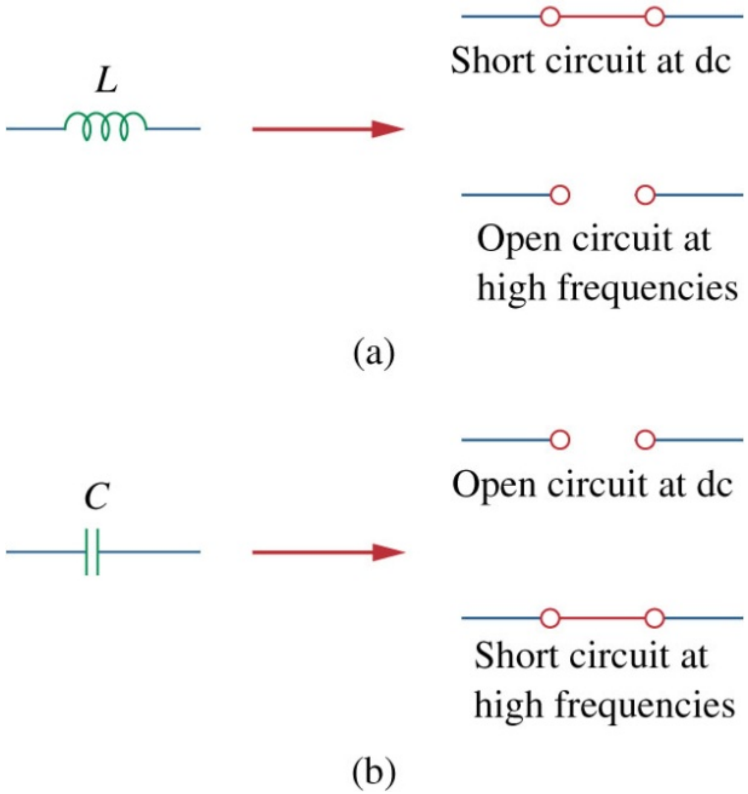
- It is possible to expand Ohm's law to capacitors and inductors.
- In time domain, this would be tricky as the ratios of voltage and current and always changing.
- But in frequency domain it is straightforward.
- The impedance of a circuit element is the ratio of the phasor voltage to the phasor current.

$$Z = \frac{V}{I} \quad \text{or} \quad V = ZI$$

- Admittance is simply the inverse of impedance.

Impedance and Admittance 2

- It is important to realize that in frequency domain, the values obtained for impedance are only valid at that frequency.
- Changing to a new frequency will require recalculating the values.
- The impedance of capacitors and inductors are shown here:



Impedance and Admittance 3

- As a complex quantity, the impedance may be expressed in rectangular form.
- The separation of the real and imaginary components is useful.
- The real part is the resistance.
- The imaginary component is called the reactance, X.
- When it is positive, we say the impedance is inductive, and capacitive when it is negative.

Impedance and Admittance 4

- Admittance, being the reciprocal of the impedance, is also a complex number.
- It is measured in units of Siemens.
- The real part of the admittance is called the conductance, *G.*
- The imaginary part is called the susceptance, *B.*
- These are all expressed in Siemens or (mhos).
- The impedance and admittance components can be related to each other:

$$G = \frac{R}{R^2 + X^2} \quad B = -\frac{X}{R^2 + X^2}$$

Impedance and Admittance 5

TABLE 9.3

Impedances and admittances of passive elements.

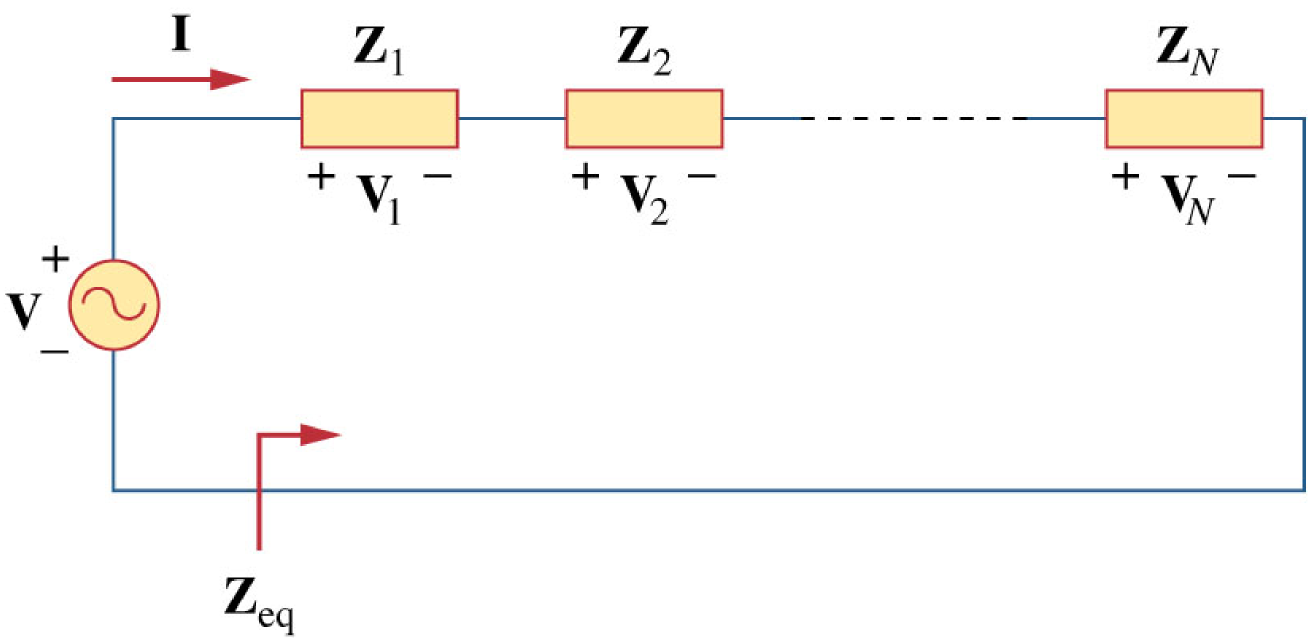
Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \frac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \frac{1}{j\omega L}$
C	$\mathbf{Z} = \frac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

Kirchoff's Laws in Frequency Domain

- A powerful aspect of phasors is that Kirchoff's laws apply to them as well.
- This means that a circuit transformed to frequency domain can be evaluated by the same methodology developed for KVL and KCL.
- One consequence is that there will likely be complex values.

Impedance Combinations 1

- Once in frequency domain, the impedance elements are generalized.
- Combinations will follow the rules for resistors:



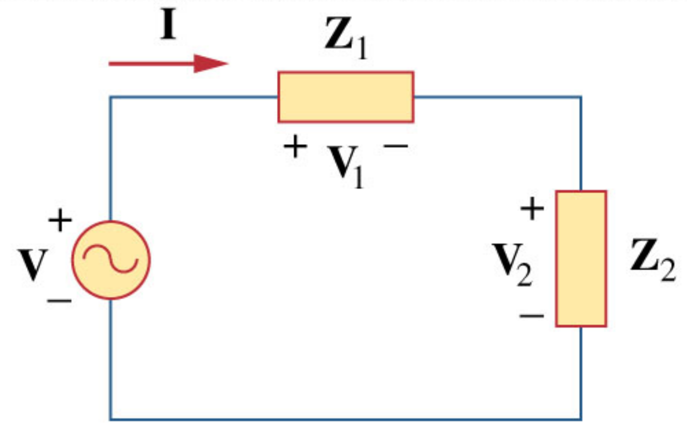
Impedance Combinations 2

- Series combinations will result in a sum of the impedance elements:

$$Z_{eq} = Z_1 + Z_2 + Z_3 + \cdots + Z_N$$

- Here then two elements in series can act like a voltage divider.

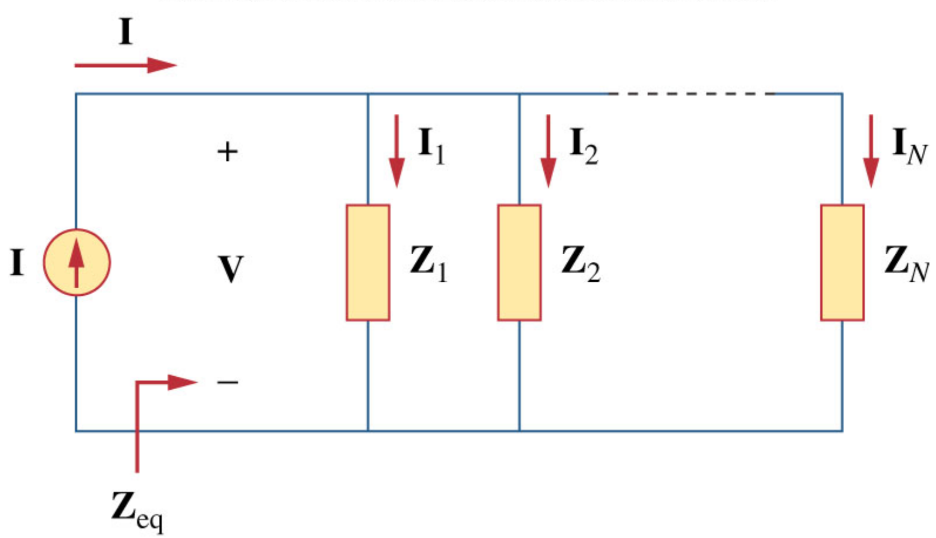
$$V_1 = \frac{Z_1}{Z_1 + Z_2} V \quad V_2 = \frac{Z_2}{Z_1 + Z_2} V$$



Parallel Combination

- Likewise, elements combined in parallel will combine in the same fashion as resistors in parallel:

$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \cdots + \frac{1}{Z_N}$$



Admittance

- Expressed as admittance, though, they are again a sum:

$$Y_{eq} = Y_1 + Y_2 + Y_3 + \cdots + Y_N$$

- Once again, these elements can act as a current divider:

$$I_1 = \frac{Z_2}{Z_1 + Z_2} I \quad I_2 = \frac{Z_1}{Z_1 + Z_2} I$$

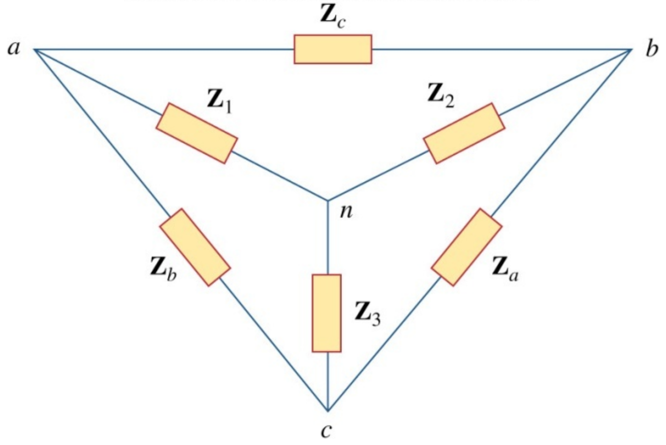
Impedance Combinations 3

- The Delta-Wye transformation is:

$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c} \qquad Z_a = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_1}$$

$$Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_c} \qquad Z_b = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_2}$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c} \qquad Z_c = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_3}$$



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