11 – Music temperament and pitch

Music sounds

Every music instrument, including the voice, uses a more or less well defined sequence of music sounds, the notes, to produce music. The notes have particular frequencies that are called „pitch“ by musicians. The frequencies or pitches of the notes stand in a particular relation to each other. There are and were different ways to build a system of music sounds in different geographical regions during different historical periods. We will consider only the currently dominating system (used in classical and pop music) that was originated in ancient Greece and further developed in Europe.

Pitch vs interval

Only a small part of the people with normal hearing are able to percieve the absolute frequency of a sound. These people, who are said to have the „absolute pitch“, can tell what key has been hit on a piano without looking at it. The human ear and brain is much more sensitive to the relations between the frequencies of two or more sounds, the music intervals. The intervals carry emotional content that makes music an art. For instance, the major third sounds joyful and affirmative whereas the minor third sounds sad.

The system of frequency relations between the sounds used in music production is called <u>music temperament</u>.

Music intervals and overtone series

Perfect music intervals (that cannot be fully achieved practically, see below) are based on the overtone series studied before in this course. A typical music sound (except of a pure sinusiodal one) consists of a fundamental frequency f_1 and its overtones:

$$f_n = nf_1, \quad n = 1, 2, 3, ...$$

Since all overtones of the same fundamental are parts of the same sound, the brain identifies all these overtones, taken separately, as different instances of the same sound. There is even the fundamental tracking phenomenon: A few overtones are sufficient to reconstruct the fundamental, so that the brain hears the actually missing fundamental when overtones are played.

Sounds with the frequencies corresponding to the overtones of a single fundamental sound great with each other and they are good candidates for music production. It would be desirable to build sequencies of music sounds based on the overtones.

Unfortunately, it turns out to be a complicated problem with no perfect solution, since too many requirements have to be satisfied at the same time. What we have as a practical solution are different systems of compromises, the currently dominating one being the so-called equal temperament using frequency relations between the sounds that only approximate the relations between the overtones.

The octave

Octave is the basic and, in a sence, trivial music interval. The frequences of the sounds forming an octave stand in the relation 2:1=2 to each other. This is the relation between the second overtone (n=2) and the fundamental (n=1).

The fifth

The frequences of the sounds forming a (perfect) fifth stand in the relation 3:2=1.5 to each other. This is the relation between the third overtone (n=3) and the second overtone (n=2).

The fourth

The frequences of the sounds forming a (perfect) fourth stand in the relation 4:3=1.3333 to each other. This is the relation between the fourth overtone (n=4) and the third overtone (n=3).

The major third

The frequences of the sounds forming a (perfect) major third stand in the relation 5:4=1.25 to each other. This is the relation between the fifth overtone (n=5) and the fourth overtone (n=4).

The minor third

The frequences of the sounds forming a (perfect) minor third stand in the relation 6:5=1.2 to each other. This is the relation between the sixth overtone (n=6) and the fifth overtone (n=5).

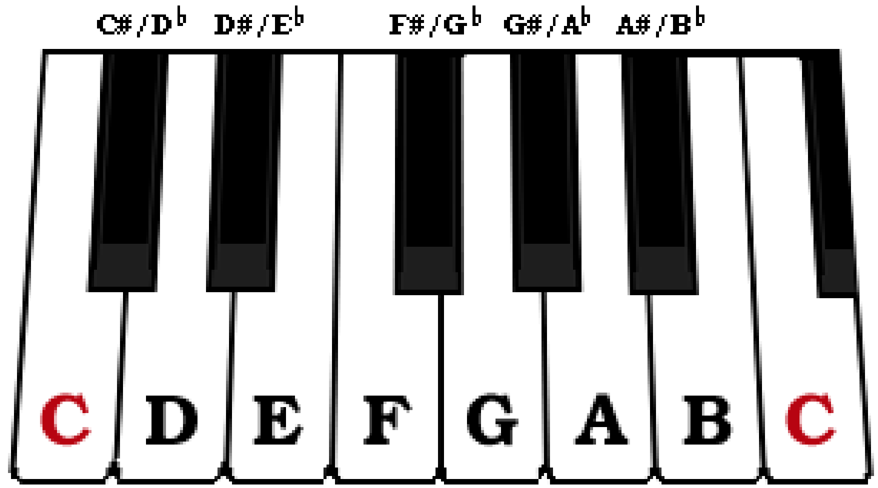
Note that the largest of these intervals is the octave and the smallest is the minor third.

Music sounds and music scales

The octave that is used as a basic building block in all temperaments is a pretty large music interval. Thus it should be divided into smaller intervals, possibly including nice-sounding intervals discussed above. The division within the octave includes the 12 smallest intervals, the so-called half-tones or semitones. Two adjaicent semitones form a tone.

In a music work or a music piece (please, do not call it „song“!) not all notes within the octave are used. On a basic music level, only 7 different notes are used, whereas the notes differing by an octave are counted as the same. Such sets of notes are called music scales. The simplest example is the C major (or simply C) scale that consists of the notes C,D,E,F,G,A,B. The two main types of scales are major and minor scales that are characterized by particular intervals (tones and semitones) between the adjaicent notes (see music theory).

In any music scale, the notes play different functions. The most important note is the so-called tonic that serves as the basis, is listed the first, and gives the name to the scale. In the C major (C) and C minor (c) scales tonic is the note „C“. One can build a music scale starting from any 12 notes in the octave, taken as the tonic. Accounting for major and minor, there are in total 2×12=24 different music scales.



We also call music scales shortly „keys“. A piece written in the C key uses the C music scale.

Choosing the temperament. Three groups of temperaments

A vital question is, as mentioned above, how to set the12 semitone intervals within the octave, that is, how to choose the music temperament. The two requirements should be satisfied:

- The intervals between the notes should possibly include the nice overtone-based intervals considered above.
- All 24 music scales should sound satisfactorily.

It turns out that these requirements cannot be satisfied at the same time, so that one has to make compromises. A great number of temperaments had been proposed during the development of music theory and practice, to variable success. All temperaments fall into three groups: Open temperaments, closed unequal temperaments and one equal temperament that dominates now.

Historically earlier open temperaments use perfect intervals (fifth, forth, etc.) in basic music scales such as C. The price to pay is that the intervals in the scales remote from C (such as the C-sharp scale) are pretty far from the perfect intervals and sound horrible. As a results, if a music instrument is tuned with an open temperament, only a limited number of music scales can be used.

The need to increase the number of usable music scales lead to the introduction of closed unequal temperaments where the imperfections of tuning (that is, deviations from the perfect intervals) are distributed all over the octave in different ways that is a too special thing to discuss here in detail. Unequal means that there are two or more different intervals close to the corresponding perfect intervals. Within this group of temperaments, music works written in different scales sound somewhat differently (that some musicians appreciate as a special „flavors“ or „colors“) whereas all of them are usable.

Equal temperament

Currently dominating temperament (that is used to tune pianos and other keyboards) is the equal temperament in which all semitones are the same and have the frequency ratio $R{:}1$ with

$$R = 2^{1/12} = 1.05946$$

Since there are 12 semitones in the octave, all 12 semitones together make up the interval

$$R^{12} = \left(2^{1/12}\right)^{12} = 2$$

that is just the octave, as it should be. As we shall see, the fifth, fourth, and thirds within the equal temperament differ a little from the corresponding perfect intervals but the imperfections are tolerable.

The equal temperament was proposed by music theorists before Bach but it was not widely used because it was more difficult to tune than the closed unequal temperaments and for other reasons. Bach appreciated the fact that within the equal temperament all music scales sound equally and he championed this temperament writing two books of his famous Well-Tempered Clavier, each book containing 24 preludes and fugues in all 24 different keys. After that the equal temperament gradually became dominating.

Frequency ratios (relative to the tonic C) for the equal temperament, compared with the perfect FRs

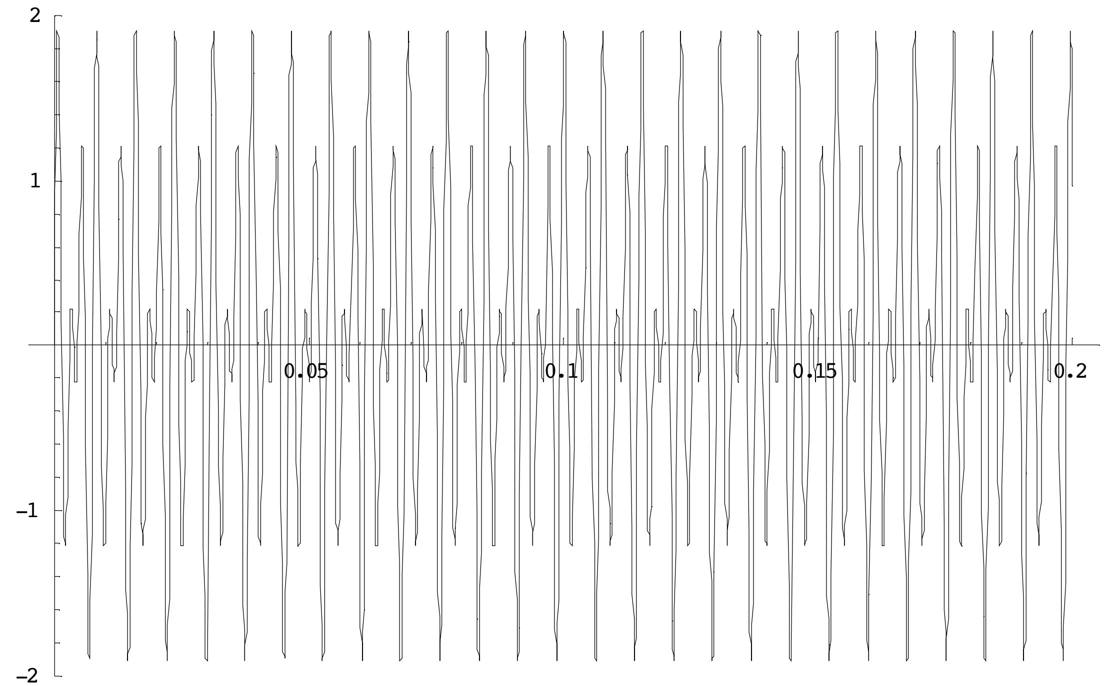
If the tonic frequency is f_T, then the frequencies of all notes f_n can be obtained by multiplying f_T by the corresponding frequency ratio:

$$f_n = f_T R^n$$

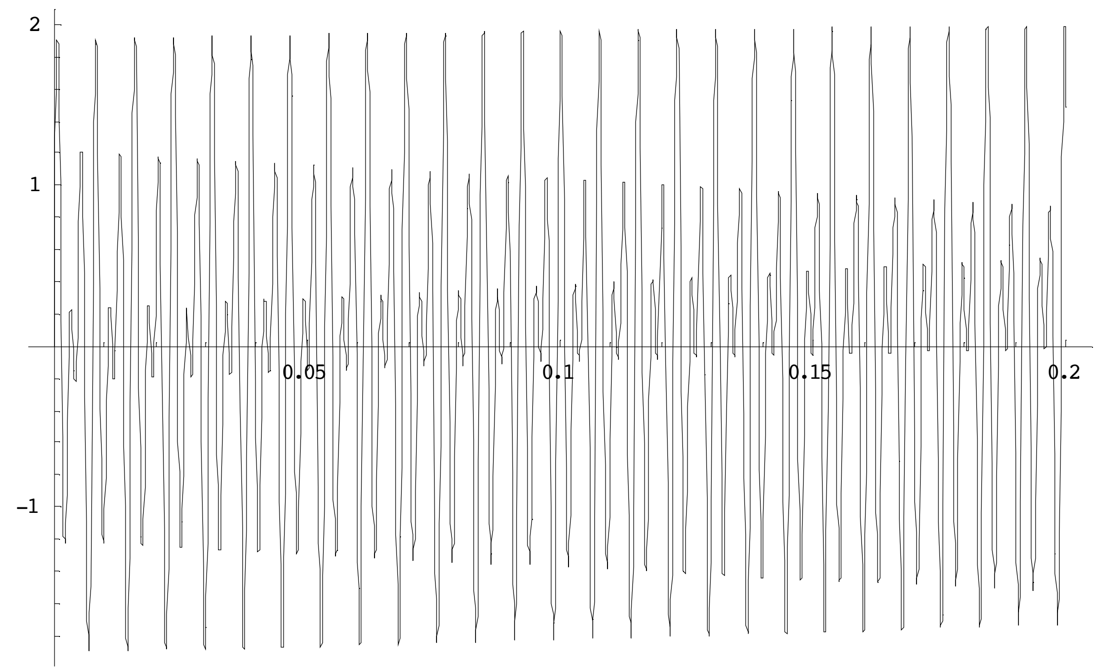
Deviations from the perfect FRs produce beats between different overtones of the two sounds that are audible and are a source of some discord. It is worth noting that these beats can be heard even in the absence of the overtones, that is, between two pure sinusoidal signals.

Interval name	*Frequency ratio*	*Perfect FR*
C - Unison	1	1
C$^\sharp$ - Minor second	R = 1.05946	
D - Major second	R^2 = 1.12246	
D$^\sharp$ - Minor third	R^3 = 1.18921	6/5 = 1.2
E - Major third	R^4 = 1.25992	5/4 = 1.25
F - Fourth	R^5 = 1.33484	4/3 = 1.33333
F$^\sharp$ - Diminished fifth	R^6 = 1.41421	
G - Fifth	R^7 = 1.49831	3/2 = 1.5
G$^\sharp$ - Minor sixth	R^8 = 1.5874	
A - Major sixth	R^9 = 1.68179	
A$^\sharp$ - Minor seventh	R^{10} = 1.7818	
B - Major seventh	R^{11} = 1.88775	
C - Octave	R^{12} = 2	2

Wave form of a perfect fifth, no beats



Wave form of an equal-tempered fifth. Beats in this pattern are seen by the eye and heard by the ear.



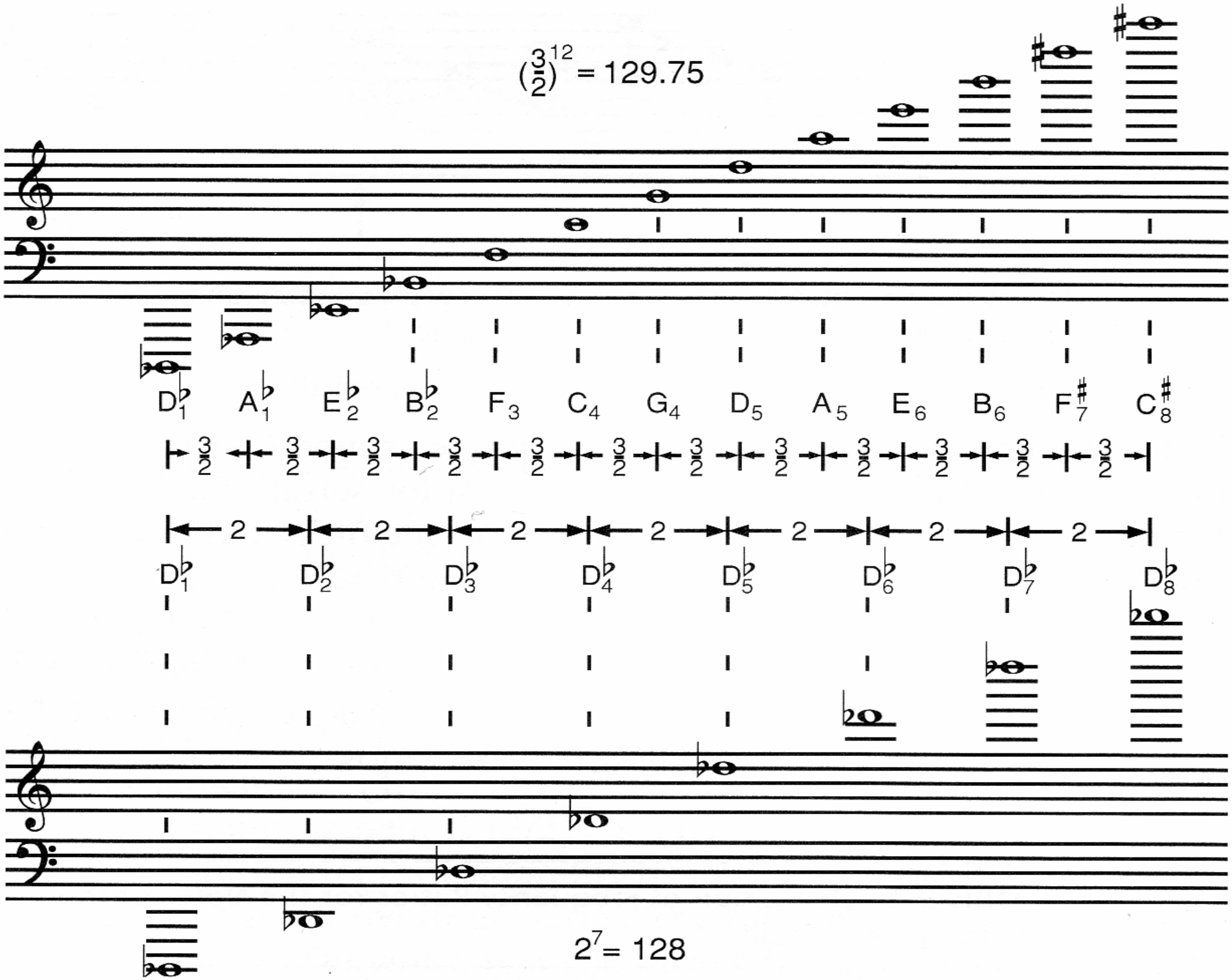
Pythagorean temperament

Pythagorean temperament was historically the first of temperaments using all 12 semitones within the octave. This temperament uses the fifth as the biulding block and tries to make all fifths perfect while preserving the octaves perfect, too.

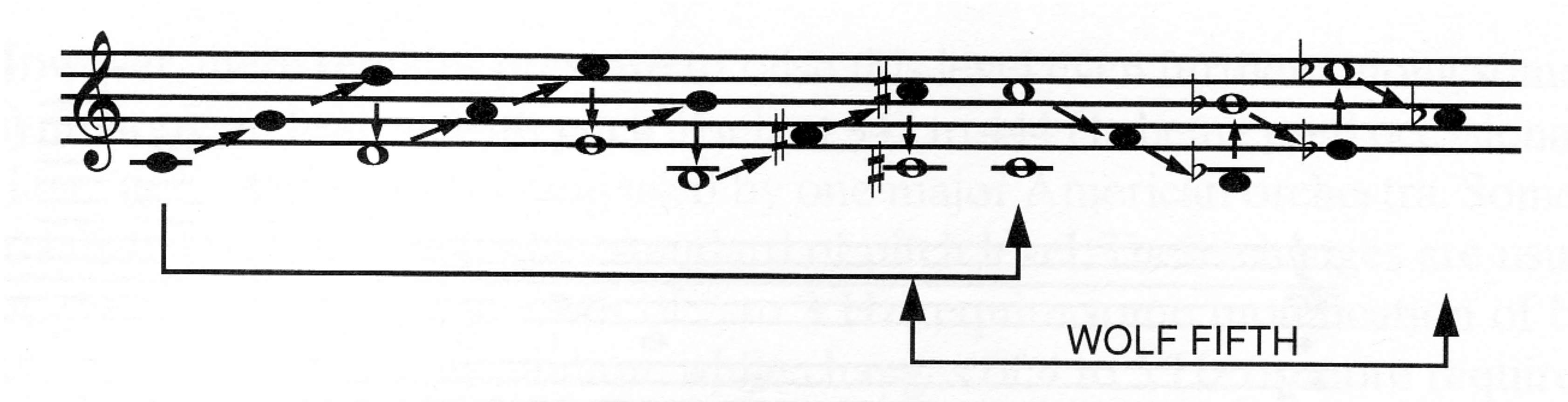
Pythagorean temperament is an open temperament, as we shall see that here all fifths but one are perfect, and the last one is very bad. This the error (of fifths) concentrates in one place.

Taking C_4 as the starting tone, one can tune the notes G_4 (up from C_4) and F_3 (down from C_4) requiring that both intervals F_3C_4 and C_4G_4 are perfect fifths and using the ear. Tuning is not very difficult since the perfect fifth is the second-simplest interval (after the octave) with the frequency ratio 3:2. Moving in this way up and down from C_4, one can tune all 12 semitons, however within 7 different octaves, from $D_1^\flat$ to $C_8^\sharp$ (see next page) This requires total 12 steps, up and down. After that all notes within all octaves can be obtained by building the octave intervals from already tuned 12 semitons.

The problem of the Pythagorean temperament is that the frequency ratio between $D_1^\flat$ and $C_8^\sharp$ equals $(3/2)^{12}=129.75$ and differs from the ratio $2^7=128$ that should be between the notes differing by 7 octaves. That is, tuning perfect fifths is incompatible with tuning perfect octaves. As tuning perfect octaves has a priority, one has to sacrifice one of the fifths. For instance, one can change the frequency of $C_8^\sharp$ by tuning it in a perfect 7-octave interval with $D_1^\flat$. But then the fifth $F_7^\sharp C_8^\sharp$ will be mistuned, the so-called „wolf fifth“. The frequency ratio of the wolf fifth is 1.47981 instead of 1.5. This fifth sounds horrible and is unusable in music. Within the Pythagorean temperament one has to use music scales not far away from C so that the notes forming the wolf fifth do not occur.


(3/2)^12 = 129.75
D♭1 A♭1 E♭2 B♭2 F3 C4 G4 D5 A5 E6 B6 F♯7 C♯8
3/2 3/2 3/2 3/2 3/2 3/2 3/2 3/2 3/2 3/2 3/2 3/2
2 2 2 2 2 2 2
D♭1 D♭2 D♭3 D♭4 D♭5 D♭6 D♭7 D♭8
2^7 = 128

The procedure of setting the Pythagorean temperament outlined above is the simplest theoretically. Practically it is impossible to tune a 7-octaves interval, as was suggested. Combining tuning perfect fifths and octaves up and down, one can achieve the goal within only a couple of octaves. One of possible realizations is shown below.



While the wolf fifth in the Pythagorean temperament can be avoided by using a limited number of keys, there is another problem of a greater practical significance. While this temperament is based on the fifths, no care is taken for other intervals. In particular, the frequency ratio of the major third CE (in any octave) is 1.26563 instead of the perfect value 5:4=1.25. The major third $C^{\#}F$ has, to the contrary, the FR 1.24859. Both of these values are further from 1.25 than the major third FR within the equal temperament 1.25992.

The problems with major thirds and the wolf fifth initiated a long research and creation of numerous improved temperaments, including closed unequal temperaments. Discussing the latter is beyond the scope of these notes.

Frequency ratios (relative to the tonic C) for the Pythagorean temperament, compared with the perfect FRs

Note that the fourth CF and many others are perfect, too.

The wolf fifth $C^{\#}G^{\#}$ has the FR $(2^7/3^4)/(3^7/2^{11})=2^{18}/3^{11}=1.47981$

Interval name	*Frequency ratio*	*Perfect FR*
C - Unison	1	1
$C^{\#}$ - Minor second	$3^7/2^{11} = 1.06787$	
D - Major second	$3^2/2^3 = 1.125$	
$D^{\#}$ - Minor third	$2^5/3^3 = 1.18519$	6/5 = 1.2
E - Major third	$3^4/2^6 = 1.26563$	5/4 = 1.25
F - Fourth	4/3 = 1.33333	4/3 = 1.33333
$F^{\#}$ - Diminished fifth	$3^6/2^9 = 1.42383$	
G - Fifth	3/2 = 1.5	3/2 = 1.5
$G^{\#}$ - Minor sixth	$2^7/3^4 = 1.58025$	
A - Major sixth	$3^3/2^4 = 1.6875$	
$A^{\#}$ - Minor seventh	$2^4/3^2 = 1.77778$	
B - Major seventh	$3^5/2^7 = 1.89844$	
C - Octave	2	2