Mathematics of Musical Temperament and Harmony

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Musical Temperament

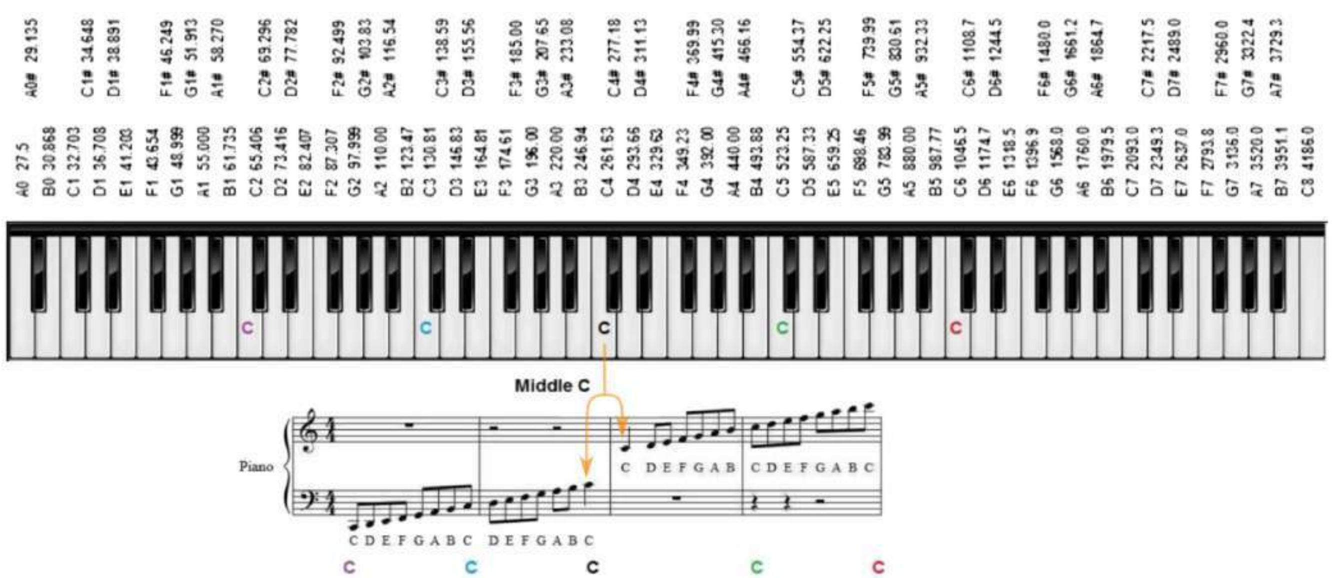
Temperament is the adjustment of intervals in tuning a piano or other musical instrument so as to fit the scale for use in different keys. Historically, the use of just intonation, Pythagorean tuning, and meantone temperament meant that such instruments could sound "in tune" in one key, or some keys, but would then have more dissonance in other keys. For example, some of these previous tuning systems were utilized that created perfect fifths, meaning that music in the keys of C, G, D, A, E, or B sounds reasonably well, but music in the keys of F#, C#, G#, or D# may sound out of tune.

An equal temperament is a musical temperament, or a system of tuning, in which the frequency interval between every pair of adjacent notes has the same ratio. In other words, the ratios of the frequencies of any adjacent pair of notes are all the same. As pitch is perceived roughly as the logarithm of frequency, equal perceived "distance" is maintained from every note to its nearest neighbor. An octave is the interval between one musical pitch and another with double its frequency. With equal temperament, an octave consists of twelve equally spaced semitones (half steps) on a logarithmic frequency scale. The equal temperament is now universal, which enables music in all key signatures to be played without any noticeable harmonic "distortion."

Even before the system was widespread, equal temperament was approximated in various degrees as a practical matter, in the small adjustments made by organ tuners and harpsichordists. The development of well temperament allowed fixed-pitch instruments to play reasonably well in all of the keys. The famous *Well-Tempered Clavier* by Johann Sebastian Bach takes full advantage of this breakthrough, with pieces written in all 24 major and minor keys. However, while unpleasant intervals (such as the wolf interval) were avoided, the sizes of intervals were still not consistent between keys, and so each key still had its own character. This variation led in the 18th century to an increase in the use of equal temperament, in which the frequency ratio between each pair of adjacent notes on the keyboard was made equal, allowing music to be transposed between keys without changing the relationship between notes. Equal temperament tuning was widely adopted in France and Germany by the late 18th century and in England by the 19th.

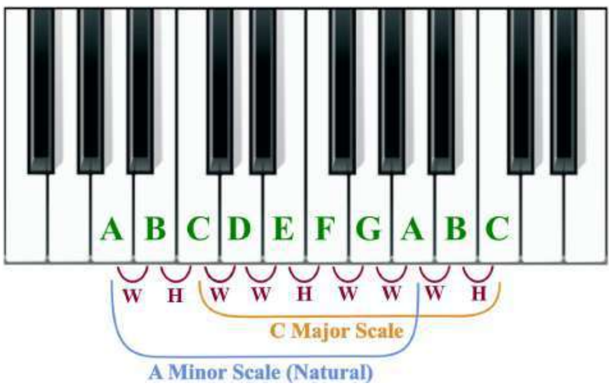
Chromatic Scale

In Western music, the chromatic scale has twelve semitones in an octave with the equal temperament. The standard piano today has 88 keys, having 7 registers and covering 7 1/3 octaves as shown below. The ideal frequency for each key is also shown with A4 = 440 Hz, the so-called concert pitch. There is only one way to construct the chromatic scale because all notes are used in a sequential manner.



Diatonic Scales

There are different diatonic scales, which are constructed from a mix of whole steps (W) and half steps (H). The major scale consists of 7 notes over an octave with the interval sequence of W–W–H–W–W–W–H. The minor scale (natural) has the interval sequence of W–H–W–W–H–W–W. The C major scale and its *relative* minor scale (Am) on the keyboard are shown below. (A pair of major and minor scales sharing the same key signature are said to be in a *relative relationship*.) Some examples of the major and minor scales are shown on the right. Other scales not listed here include the modal scales.



			Tonic	Supertonic	Mediant	Subdominant	Dominant	Submediant	Leading tone
Major scale	Step		W	W	H	W	W	W	H
	Key example	**C**	C	D	E	F	G	A	B
		G	G	A	B	C	D	E	F#
		D	D	E	F#	G	A	B	C#
		A	A	B	C#	D	E	F#	G#
		E	E	F#	G#	A	B	C#	D#
Minor scale (natural)	Step		W	H	W	W	H	W	W
	Key example	**Am**	A	B	C	D	E	F	G
		Em	E	F#	G	A	B	C	D
		Bm	B	C#	D	E	F#	G	A
		F#m	F#	G#	A	B	C#	D	E
		C#m	C#	D#	E	F#	G#	A	B
Minor scale (harmonic)	Step		W	H	W	W	H	W+	H
	Key example	**Am**	A	B	C	D	E	F	G#
		Em	E	F#	G	A	B	C	D#
		Bm	B	C#	D	E	F#	G	A#
		F#m	F#	G#	A	B	C#	D	E#
		C#m	C#	D#	E	F#	G#	A	B#
Minor scale (melodic) *	Step		W	H	W	W	W	W	H
	Key example	**Am**	A	B	C	D	E	F#	G#
		Em	E	F#	G	A	B	C#	D#
		Bm	B	C#	D	E	F#	G#	A#
		F#m	F#	G#	A	B	C#	D#	E#
		C#m	C#	D#	E	F#	G#	A#	B#

* The descending of the melodic minor scale follows the interval sequence of the natural minor scale.

Geometric Series

A geometric series is a series with a constant ratio between successive terms. With the equal temperament, the frequencies of the chromatic scale form a geometric series. The series is completely characterized by only one parameter, the frequency ratio between adjacent semitones. So, let's determine this ratio *r*. We start with an arbitrary note of frequency f_0. The next 12 semitones in the octave form the series $\{f_0, f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12}\}$, where $f_{12} = 2 \times f_0$. The geometric series is

$$\{f_0,\ f_0 r,\ f_0 r^2,\ f_0 r^3,\ f_0 r^4,\ f_0 r^5,\ f_0 r^6,\ f_0 r^7,\ f_0 r^8,\ f_0 r^9,\ f_0 r^{10},\ f_0 r^{11},\ f_0 r^{12}\}$$

The octave relationship results in: $f_{12} = 2 \times f_0 = f_0 r^{12} \Rightarrow 2 = r^{12}$

Take the 12th root on both sides, we have: $r = \sqrt[12]{2} = 2^{\frac{1}{12}} = 1.059463094$.

The interval of two adjacent notes is further divided into 100 cents. The ratio between two frequencies separated by 1 cent is $r_{cent} = 2^{\frac{1}{1200}} = 1.00057779$.

Now, we determine the difference between two frequencies in terms of cents. Let

$f_1 = f_0 2^{c_1/1200}$ and $f_2 = f_0 2^{c_2/1200}$. The ratio of the two frequencies is:

$$\frac{f_2}{f_1} = \frac{f_0 2^{c_2/1200}}{f_0 2^{c_1/1200}} = 2^{(c_2/1200 - c_1/1200)} = 2^{(c_2 - c_1)/1200} \quad \Rightarrow \quad c_2 - c_1 = 1200 \ \log_2 \frac{f_2}{f_1}$$

Exponential and Logarithm

Here is a side note on the mathematics of exponential and logarithmic functions. An exponential function is given by the general form:

$y = b^x$, where b is the base and x is the exponent.

The base is usually a number > 1. Commonly encountered bases include 2, 10, and e, where e is the so-called Euler's number $e = 2.71828182845904523\ldots$.

The logarithm is the inverse function (or anti-function) of the exponential.

$x = \log_b y$

Here are some examples:

$1 = 10^0$	$0 = \log_{10} 1 = \log_{10} 10^0$	$1 = 2^0$	$0 = \log_2 1 = \log_2 2^0$
$10 = 10^1$	$1 = \log_{10} 10 = \log_{10} 10^1$	$2 = 2^1$	$1 = \log_2 2 = \log_2 2^1$
$100 = 10^2$	$2 = \log_{10} 100 = \log_{10} 10^2$	$4 = 2^2$	$2 = \log_2 4 = \log_2 2^2$
$1000 = 10^3$	$3 = \log_{10} 1000 = \log_{10} 10^3$	$8 = 2^3$	$3 = \log_2 8 = \log_2 2^3$
$10000 = 10^4$	$4 = \log_{10} 10000 = \log_{10} 10^4$	$16 = 2^4$	$4 = \log_2 16 = \log_2 2^4$

Some properties of exponentials and logarithms are summarized below.

Operation	Laws of exponentials	Laws of logarithms
multiplication	$b^\alpha \times b^\beta = b^{(\alpha+\beta)}$	$\log(\alpha \times \beta) = \log\alpha + \log\beta$
division	$b^\alpha \,/\, b^\beta = b^{(\alpha-\beta)}$	$\log(\alpha \,/\, \beta) = \log\alpha - \log\beta$
exponentiation	$(b^\alpha)^\beta = b^{\alpha\beta}$	$\log\alpha^\beta = \beta \log\alpha$
zero property	$b^0 = 1$	$\log 1 = 0$
inverse	$b^{-1} = 1/b$	$\log\alpha^{-1} = \log(1/\alpha) = -\log\alpha$
nth root	$\sqrt[n]{\alpha} = \alpha^{1/n}$	$\log\alpha^{1/n} = \log\alpha \,/\, n$

In summary, the cent is a logarithmic unit of measure used for musical intervals. Twelve-tone equal temperament divides the octave into 12 semitones of 100 cents each. In other words, an octave spans over 1,200 cents. Typically, cents are used to express small intervals, or to compare the sizes of comparable intervals in different tuning systems. It is difficult to establish how many cents are perceptible to humans; this accuracy varies greatly from person to person and depends on the frequency, the amplitude, and the timbre (tone quality). Normal adults are able to recognize pitch differences of as small as 25 cents very reliably. An online test to determine your pitch perception at 500 Hz is available at <http://jakemandell.com/adaptivepitch/>.

Example 1. *Concert pitch* is a standard for tuning of musical instruments, internationally agreed upon in 1960, in which the note A above middle C (A4) has a frequency of 440 Hz. Based on equal temperament, determine the frequency for middle C (C4).

From the keyboard shown on the previous page, C4 is 9 semitones below A4. Thus we have

$$\frac{f_{A4}}{f_{C4}} = 2^{9/12} = 2^{0.75} = 1.6818 \quad \Rightarrow \quad f_{C4} = \frac{f_{A4}}{1.6818} = \frac{440}{1.6818} = 261.6 \text{ Hz.}$$

Example 2. With C4 = 261.6 Hz, determine the frequency of perfect fifth (G4) based on equal temperament and harmonic fraction, respectively. Based on harmonic fraction, the frequency of G4 should be 3/2 of that of C4.

Based on Harmonic fraction, $f_{G4} = \dfrac{3}{2} f_{C4} = 1.5 \times 261.6 = 392.4$ Hz.

Based on equal temperament, $\dfrac{f_{G4}}{f_{C4}} = 2^{7/12} = 1.4983 \quad \Rightarrow \quad f_{G4} = 261.6 \times 1.4983 = 392.0$ Hz.

The frequency difference is 392.4 – 392.0 = 0.4 Hz

Example 3. For the above problem, what's the frequency difference in cents?

$$c_2 - c_1 = 1200 \ \log_2 \frac{f_2}{f_1} = 1200 \ \log_2 \frac{392.4}{392.0} = 1.77 \simeq 2 \ \text{cents}$$

Example 4. Repeat the above problems for major third (E4). Based on harmonic fraction, the frequency of E4 should be 5/4 of that of C4.

Based on Harmonic fraction, $f_{E4} = \dfrac{5}{4} f_{C4} = 1.25 \times 261.6 = 327.0$ Hz.

Based on equal temperament, $\dfrac{f_{E4}}{f_{C4}} = 2^{4/12} = 1.2599 \quad \Rightarrow \quad f_{E4} = 261.6 \times 1.2599 = 329.6$ Hz.

The frequency difference is 329.6 – 327.0 = 2.6 Hz

$$c_2 - c_1 = 1200 \ \log_2 \frac{f_2}{f_1} = 1200 \ \log_2 \frac{329.6}{327.0} = 14 \ \text{cents.}$$

Note: This computation can be done on the OpenOffice spreadsheet: =log(329.6/327.0; 2)

Example 5. The 1st violin is tuned to A4 = 440 Hz, the 2nd violin A4 = 435 Hz, and the 3nd violin A4 = 442 Hz. What are the pitch differences among them in terms of cents?

The 2nd violin is lower than the 1st violin by $c_1 - c_2 = 1200 \ \log_2 \dfrac{440}{435} = 20$ cents.

The 3rd violin is higher than the 1st violin by $c_3 - c_1 = 1200 \ \log_2 \dfrac{442}{440} = 8$ cents.

The 3rd violin is higher than the 2nd violin by $c_3 - c_2 = 1200 \ \log_2 \dfrac{442}{435} = 28$ cents.

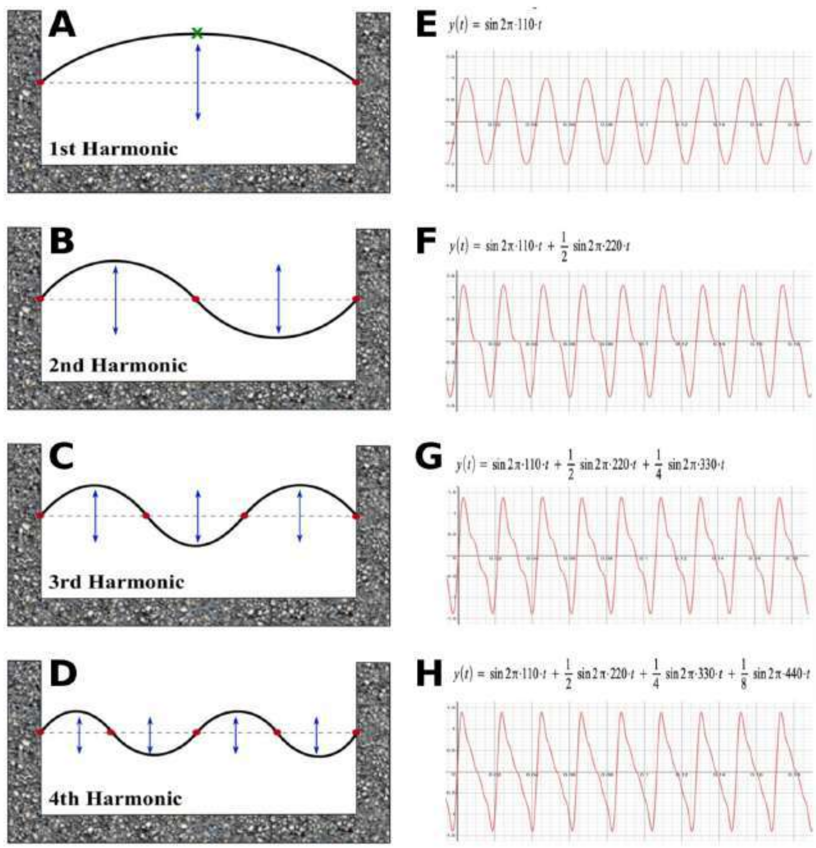
Notice that, once converted to cents, the frequency differences are additive. This follows the multiplication law of logarithms shown on the previous page.

Frequency ratio: $(\dfrac{f_3}{f_1})(\dfrac{f_1}{f_2}) = \dfrac{f_3}{f_2} = (\dfrac{442}{440})(\dfrac{440}{435}) = \dfrac{442}{435}$

Frequency difference: $(c_3 - c_1) + (c_1 - c_2) = c_3 - c_2 = 8 \ \text{cents} + 20 \ \text{cents} = 28 \ \text{cents}$

Vibrations

As shown in panel A of the figure below, a string is fixed at two end points (called nodes) and is plucked. The simplest vibration can be characterized by a sine wave. Plotting the displacement at the middle point of the string (marked by the green X), we have $y(t) = A \sin 2\pi f_0 t$, where f_0 is the frequency of the vibration and A is the amplitude. The amplitude will decay and the vibration will eventually cease. But let's ignore the decay for now and assume A is a constant. The fundamental frequency f_0, also called the 1st harmonic, depends on the length, tension, and weight of the string. The system also supports vibrations at frequencies that are integer multiples of f_0. As shown in panel B, the 2nd harmonic vibrates at the frequency $2 f_0$. An additional node occurs at the middle point of the string, which remains stationary. Similarly, the 3rd harmonic with the frequency of $3 f_0$ and the 4th harmonic with the frequency of $4 f_0$, respectively, are shown in panel C and D. These harmonics (vibration modes) co-exist and jointly determine the waveform of the vibration. Because the nodes at the two ends are fixed, frequencies other than the harmonic frequencies can not exist.



To demonstrate how the harmonics affect the waveform, we perform a mathematical simulation as follows. Let's assume the 5th string of a guitar is plucked, which is tuned at A2 = 110 Hz. The 1st harmonic with the amplitude A set to 1 is given by:

$$y(t) = \sin 2\pi \cdot 110 \cdot t$$

The waveform is generated with an online graphing calculator <www.desmos.com/calculator>, as shown in panel E. Next, we add the 2nd harmonic with an amplitude of 1/2. The resulting waveform is shown in panel F.

$$y(t) = \sin 2\pi \cdot 110 \cdot t + \frac{1}{2} \sin 2\pi \cdot 220 \cdot t$$

Panel G shows the waveform with the inclusion of 1/4 of the 3rd harmonic.

$$y(t) = \sin 2\pi \cdot 110 \cdot t + \frac{1}{2} \sin 2\pi \cdot 220 \cdot t + \frac{1}{4} \sin 2\pi \cdot 330 \cdot t$$

Finally, panel H shows waveform with the inclusion of 1/8 of the 4th harmonic.

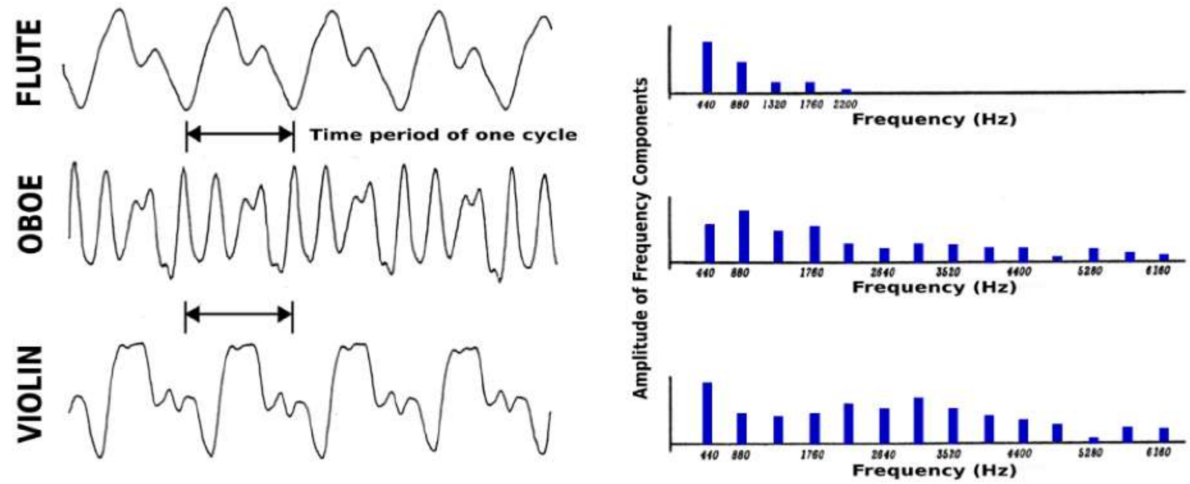
$$y(t) = \sin 2\pi \cdot 110 \cdot t + \frac{1}{2} \sin 2\pi \cdot 220 \cdot t + \frac{1}{4} \sin 2\pi \cdot 330 \cdot t + \frac{1}{8} \sin 2\pi \cdot 440 \cdot t$$

The resulting waveform is now more triangular in shape than sinusoidal.

In summary, the system of a vibrating string supports an ensemble of harmonics with integer multiples of the fundamental frequencies, but no other frequencies. The harmonics change the shape of the vibration waveform, which affect the tone quality (timbre) of the resulting sound.

Harmonics and Timbre

Figure below shows recorded waveforms from three instruments (flute, oboe, and violin) playing the A4 note (440 Hz) and their harmonic components. The flute is an aerophone or reedless wind instrument. The oboe is a double reed woodwind instrument. The violin is a string instrument. Although all three instruments play the same note, the harmonic contents are quite different to give each instrument a unique timbre (tone quality).

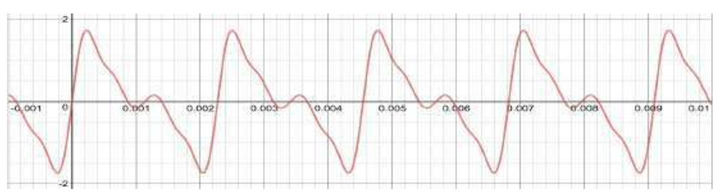


The time period of one cycle shown in the figure is the reciprocal of the fundamental frequency:

$$T = \frac{1}{f_0} = \frac{1}{440 \ \text{Hz}} = 0.00227 \ \text{s}$$

We now use the graphing calculator technique from the previous section to simulate the waveform of the flute. The resulting waveform is shown below.

$$y(t) = \sin 2\pi \cdot 440 \cdot t + \frac{4}{5} \sin 2\pi \cdot 880 \cdot t + \frac{1}{4} \sin 2\pi \cdot 1320 \cdot t + \frac{1}{4} \sin 2\pi \cdot 1760 \cdot t + \frac{1}{20} \sin 2\pi \cdot 2200 \cdot t$$



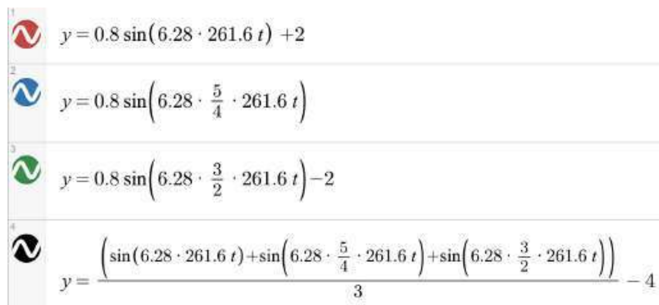
While the simulated waveform bears the general shape of the actual waveform, there is some degree of discrepancy. This may be due to unrepresented *phase* components, which are time delays among the different harmonics. The effect of the phase will be further discussed later.

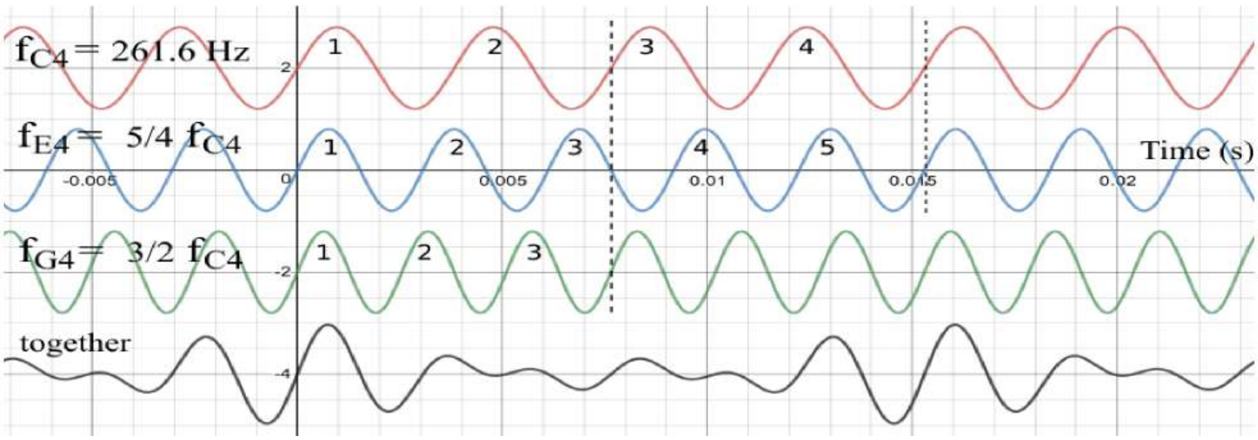
Just intonation

Just intonation or pure intonation is the tuning musical intervals as small integer ratios of frequencies. Any interval tuned in this way is called a just interval. In just intonation the diatonic scale may be easily constructed using the three simplest intervals within the octave, the perfect fifth (3/2), perfect fourth (4/3), and the major third (5/4). As forms of the fifth and third are naturally present in the overtone series of harmonic resonators, this is a very simple process. The table shows the harmonic fractions between the frequencies of the just intonation for the C major scale.

C	D	E	F	G	A	B	C
1	9/8	5/4	4/3	3/2	5/3	15/8	2

An example is generated by using an online graphing calculator <https://www.desmos.com/calculator>. The note C4 is represented by a pure sine wave at 261.6 Hz. Based on the harmonic fractions, the note E4 is 5/4 times higher and the note G4 is 3/2 times higher than C4. Together the three waves form a stable, periodic oscillation. The equations entries are shown on the right with the waveforms shown below.



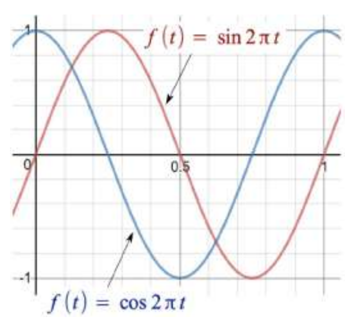


Fourier Analysis

To probe further, the representation of a periodic signal by its harmonics was first studied by the French mathematician and physicist Jean-Baptiste Joseph Fourier (1768–1830). The Fourier series analysis was initially concerned with periodic signals. It was later expanded to non-periodic signals by using the Fourier transform. The Fourier transform has many theoretical and practical applications, which provide the foundation for areas such as linear systems and signal processing.

Jean-Baptiste Joseph Fourier (1768–1830, France)

A Fourier series is a way to represent a periodic function as the weighted sum of simple oscillating functions, namely sines and cosines. Why sines and cosines? The answer is related to the phase component (time delay) mentioned previously. As shown by the figure on the right, the sine and the cosine form a so-called orthogonal basis; They are separated by a phase angle of 90 degrees ($\pi/2$). Any angle can be represented by a linear combination of them. By definition a linear combination of sine and cosine is $a \sin 2\pi t + b \cos 2\pi t$, where a and b are constants.



In the following, we will present the formulas for the Fourier series, which utilize the notions of calculus and complex variables. If you don't have these mathematical background,s it's quite alright and please just try to follow the notations.

It is somewhat cumbersome to carry the coefficients for both sine and cosine. Thus, we introduce the complex exponential from another important mathematician Leonhard Euler. The famous Euler's number e is an irrational number: e = 2.71828182845904523 (and more). The Euler's formula represents sine and cosine with a complex exponential:

$$e^{jx} = \cos x + j \sin x \text{, where } j = \sqrt{-1} .$$

A special case of the above formula is known as Euler's identity:

$$e^{j\pi} + 1 = 0 .$$

These relationships are illustrated with the unit circle as shown in the figure at the lower-right.

Leonhard Euler (1707–1783, Switzerland)

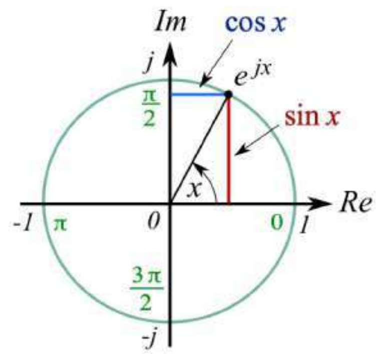
Finally, we present the Fourier series. A *time-domain* periodic signal $f(t)$ with the fundamental frequency of f_0 can be represented by a linear combination of complex exponentials,:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn2\pi f_0 t}$$

The Fourier coefficients c_n specify the weight on each harmonic in the *frequency-domai*n. The Fourier coefficients are computed according to:

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jn2\pi f_0 t} dt ,$$

where $T = 1/f_0$ is the period of the signal.



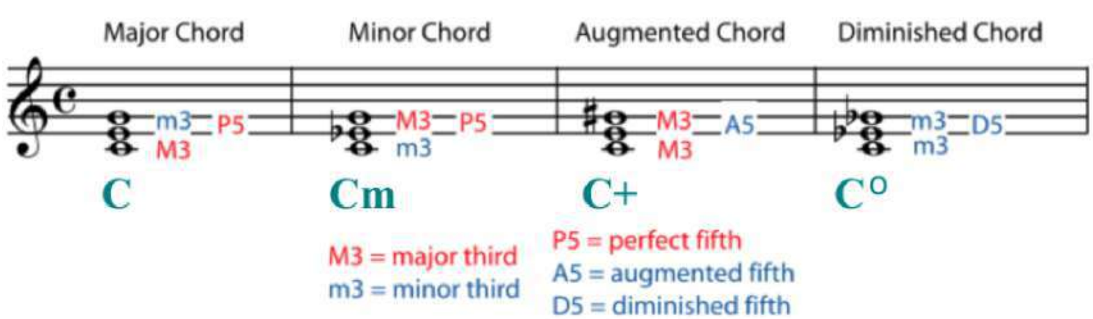
Harmony

In music, harmony considers the process by which the composition of individual sounds, or superpositions of sounds, is analyzed by hearing. Usually, this means simultaneously occurring frequencies, pitches, or chords. The study of harmony involves chords and their construction and chord progressions and the principles of connection that govern them. Harmony is often said to refer to the "vertical" aspect of music, as distinguished from melodic line, or the "horizontal" aspect [Wikipedia].

A <u>chord</u> is a group of three or more notes sounded together, as a basis of harmony. A <u>triad</u> is a a three-note chord consisting of:

- the root – this note specifying the name of the chord;
- the third – its interval above the root being a minor third (3 semitones) or a major third (4 semitones);
- the fifth – its interval above the third being a minor third or a major third.

With the choice of minor third and major third for two intervals, there are a total of 4 possible combinations. Using C as the root note, the four chords are shown below.



The diagrams below show all the common triads belong to each major keys (left chart) and minor keys (right chart). Roman numerals indicate each chord position relative to the scale.

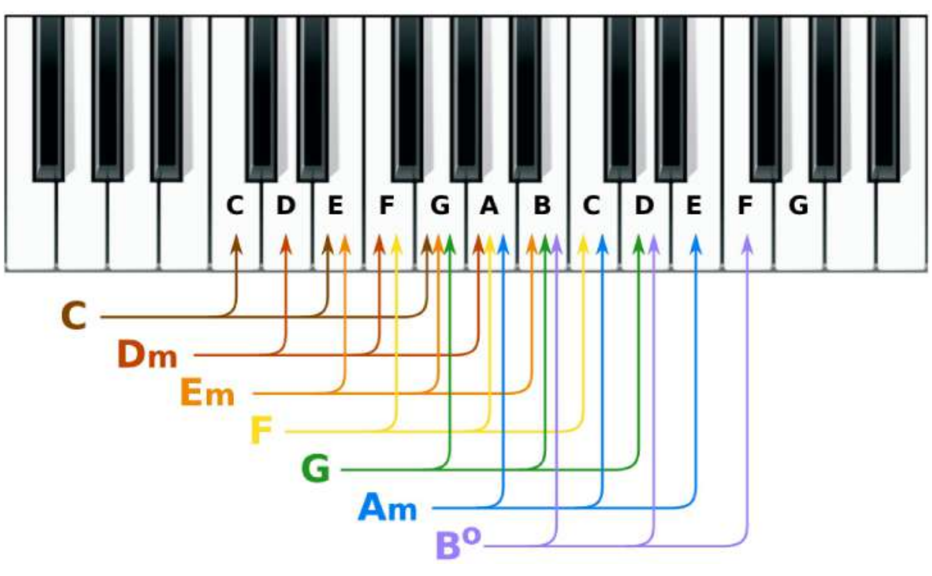
Chords In All Major Keys

Major Keys	I	ii	iii	IV	V	vi	vii°
C	C	Dm	Em	F	G	Am	B°
C♯	C♯	D♯m	E♯m	F♯	G♯	A♯m	B♯°
D♭	D♭	E♭m	Fm	G♭	A♭	B♭m	C°
D	D	Em	F♯m	G	A	Bm	C♯°
E♭	E♭	Fm	Gm	A♭	B♭	Cm	D°
E	E	F♯m	G♯m	A	B	C♯m	D♯°
F	F	Gm	Am	B♭	C	Dm	E°
F♯	F♯	G♯m	A♯m	B	C♯	D♯m	E♯°
G♭	G♭	A♭m	B♭m	C♭	D♭	E♭m	F°
G	G	Am	Bm	C	D	Em	F♯°
A♭	A♭	B♭m	Cm	D♭	E♭	Fm	G°
A	A	Bm	C♯m	D	E	F♯m	G♯°
B♭	B♭	Cm	Dm	E♭	F	Gm	A°
B	B	C♯m	D♯m	E	F♯	G♯m	A♯°

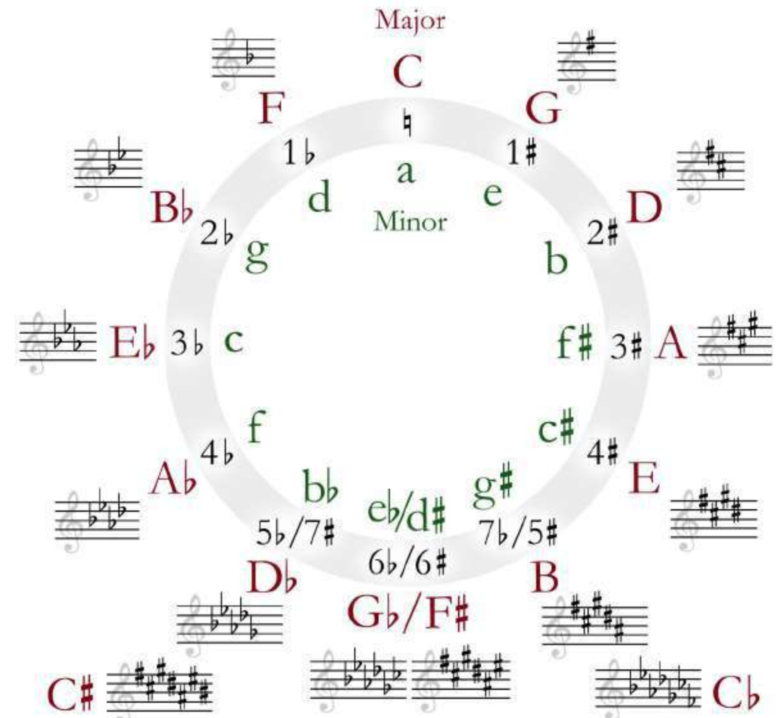
Chords In All Minor Keys

Minor Keys	i	ii°	III	iv	V	VI	VII
Cm	Cm	D°	E♭	Fm	G	A♭	B♭
C♯m	C♯m	D♯°	E	F♯m	G♯	A	B
Dm	Dm	E°	F	Gm	A	B♭	C
D♯m	D♯m	E♯°	F♯	G♯m	A♯	B	C♯
E♭m	E♭m	F°	G♭	A♭m	B♭	C♭	D♭
Em	Em	F♯°	G	Am	B	C	D
Fm	Fm	G°	A♭	B♭m	C	D♭	E♭
F♯m	F♯m	G♯°	A	Bm	C♯	D	E
Gm	Gm	A°	B♭	Cm	D	E♭	F
G♯m	G♯m	A♯°	B	C♯m	D♯	E	F♯
A♭m	A♭m	B♭°	C♭	D♭m	E♭	F♭	G♭
Am	Am	B°	C	Dm	E	F	G
A♯m	A♯m	B♯°	C♯	D♯m	E♯	F♯	G♯
B♭m	B♭m	C°	D♭	E♭m	F	G♭	A♭
Bm	Bm	C♯°	D	Em	F♯	G	A

The figure on the right demonstrates how the triads are played on a keyboard and how the different types of chords are formed. Using the C major key as an example, a triad is played with three fingers. usually the thumb, the middle finger, and the pinky. The first chord is C major: a major third (4 semitones) between C and E, and a minor third (3 semitones) between E and G. The next chord is D minor: a minor third between D and F, and a major third between F and A. This process continues until the B diminished chord: a minor third between B and D, and another minor third between D and F.



The equal temperament tuning system was developed after Bach's time. Bach composed the Well-Tempered Clavier as a departure from the various meantone tunings that were used in earlier music. Bach's motivation was to demonstrate the varying key colors in well tempered tuning as one progresses around the circle of fifths. The circle of fifths as shown in the figure is the relationship among the 12 tones of the chromatic scale, their corresponding key signatures, and the associated major and minor keys. More specifically, it is a geometrical representation of relationships among the 12 pitch classes of the chromatic scale in pitch class space.



In Bach's time there were no recording devices nor frequency measurement instruments. Therefore, we will never know exactly how Bach tuned his harpsichord to play the Well -Tempered Clavier. In 1799, Thomas Young published his version of the well temperament tuning.

Equal temperament tuning is ubiquitous nowadays. The twelve-tone serialism, initiated by the Austrian composer Arnold Schoenberg (1874–1951), emphasizes that all 12 notes of the chromatic scale are sounded as often as one another in a piece of music while preventing the emphasis of any one note through the use of tone rows, orderings of the 12 pitch classes. All 12 notes are thus given more or less equal importance. Because the music avoids being in a key, the twelve-tone serialism unquestionably favors the equal temperament tuning.

However, some people argue that equal temperament is not necessarily the best choice in order to bring out the key colors, especially for early music. See notes of Prof. Michael Rubinstein of the University of Waterloo <http://www.math.uwaterloo.ca/ ~mrubinst/tuning/ tuning.html>.

From the point of view of physics, the harmony is best formed when the frequencies of the notes are related by exact integer fractions. For example, the frequency of the perfect fifth should be 3/2 of the root note frequency. Thus, the nodes of vibrations will meet up every second cycle of the root node and every third cycle of the fifth. The resulting waveform is periodical, stable, and sounding in harmony.

To provide a quantitative analysis for the aforementioned discussion, we now compute the frequencies of the chromatic scale from C4 to C5 using the equal temperament tuning and the well temperament tuning. The Harmonic Fraction is compared to because it should provide the best harmony. To demonstrate how the computation is done, let's use E4 (major third) as an example. As A4 is tuned to 440 Hz, the frequency for C4 is 261.6 Hz.

Base on harmonic fraction (HF): $f_{E4} = (5/4) \times f_{C4} = 1.25 \times 261.6 = 327$ Hz.

Base on equal temperament (ET): $f_{E4} = (2^{5/12}) \times f_{C4} = 1.26 \times 261.6 = 329.6$ Hz.

Base on well temperament (WT): $f_{E4} = 1.2539 \times f_{C4} = 1.2539 \times 261.6 = 328$ Hz.

Difference between ET and HF: $329.6 - 327 = 2.6$ Hz, or $1200 \log_2(329.6/327) = 13.7$ ¢.

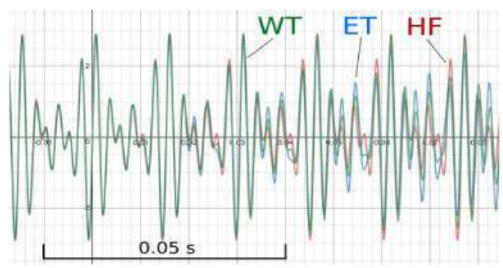
Difference between WT and HF: $328 - 327 = 1.0$ Hz, or $1200 \log_2(328/327) = 5.4$ ¢.

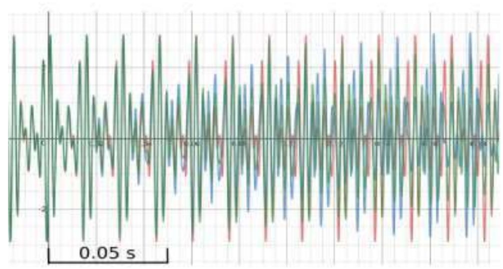
A comparison among harmonic fraction (HF), equal temperament (ET), and well temperament (WT) for the 4th octave is shown below. The spreadsheet for generating the numbers can be downloaded from the course webpage.

Interval name	Note name	No. of semitones	Harmonic fraction	Harmonic series	Harmonic octave 4 frequencies (Hz)	Equal temperament	ET octave 4 frequencies (Hz)	Difference (Hz)	Difference (cents)	Thomas Young well-temperament*	WT octave 4 frequencies (Hz)	Difference (Hz)	Difference (cents)	Remark
						1.0595								2^(1/12)
perfect unison	C	0	1/1	1.000	261.6	1.000	261.6	0.0	0.0	1.000	261.6	0.0	0.0	Middle C
minor second	C#	1	15/14	1.071	280.3	1.059	277.2	-3.1	-19.4	1.056	276.2	-4.1	-25.6	
major second	D	2	9/8	1.125	294.3	1.122	293.7	-0.7	-3.9	1.120	293.0	-1.4	-8.1	
minor third	D#	3	6/5	1.200	314.0	1.189	311.1	-2.8	-15.6	1.1877	310.7	-3.2	-17.8	
major third	E	4	5/4	1.250	327.0	1.260	329.6	2.6	13.7	1.2539	328.0	1.0	5.4	Major third
perfect fourth	F	5	4/3	1.333	348.8	1.335	349.2	0.4	2.0	1.3347	349.2	0.4	1.8	
tritone	F#	6	7/5	1.400	366.3	1.414	370.0	3.7	17.5	1.4076	368.3	2.0	9.4	
perfect fifth	G	7	3/2	1.500	392.4	1.498	392.0	-0.4	-2.0	1.4965	391.5	-0.9	-4.0	Perfect fifth
minor sixth	G#	8	8/5	1.600	418.6	1.587	415.3	-3.3	-13.7	1.5836	414.3	-4.3	-17.8	
major sixth	A	9	5/3	1.667	436.0	1.682	440.0	4.0	15.6	1.6757	438.4	2.4	9.4	A4 = 440
minor seventh	A#	10	9/5	1.800	470.9	1.782	466.2	-4.8	-17.6	1.7815	466.1	-4.8	-17.8	
major seventh	B	11	15/8	1.875	490.5	1.888	493.9	3.3	11.7	1.8788	491.6	1.0	3.5	
perfect octave	C	12	2/1	2.000	523.3	2.000	523.3	0.0	0.0	2.000	523.3	0.0	0.0	

* Reference: Well vs. Equal Temperament http://www.math.uwaterloo.ca/~mrubinst/tuning/tuning.html

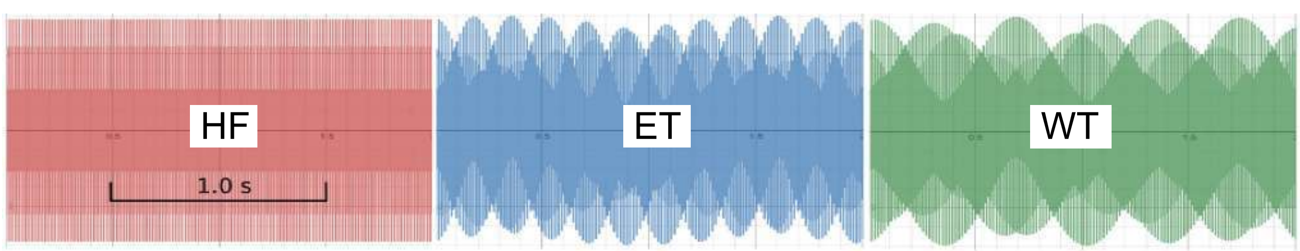
Hear the difference (Bach's Bb minor Prelude from the Well Tempered Clavier):
https://www.youtube.com/watch?v=6OxXE3GLgJk





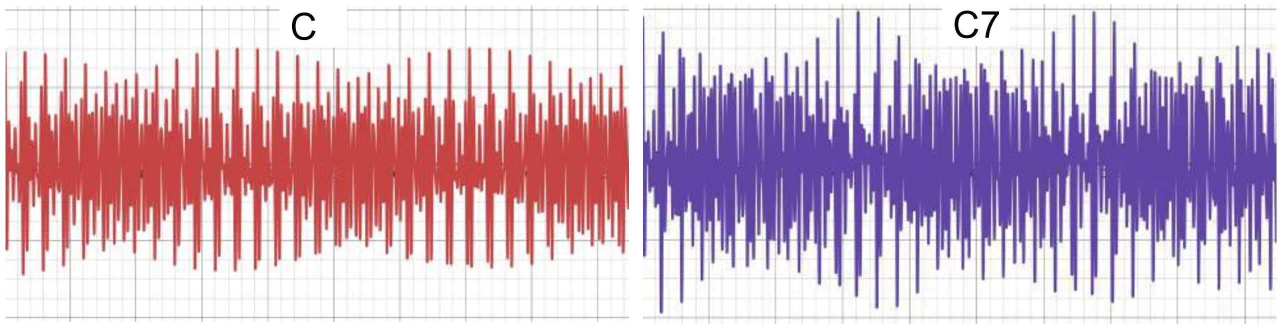
The above table shows how each individual note in the chromatic scale is in harmony with C4. The C major chord consists of C4, E4, and G4. With equal temperament, G4 is only off by 2 cents, whereas E4 is off by 14 cents. With well temperament, G4 is off by 4 cents, and E4 is off by 5 cents. Thus, for the C major chord well temperament tuning should sound more in harmony than the equal temperament.

Using the graphing calculator, the waveforms of harmonic fraction (HF), equal temperament (ET), and well temperament (WT) are plotted: $y(t)=\sin 2\pi f_{C4} t+\sin 2\pi f_{E4} t+\sin 2\pi f_{G4} t$. The waveforms of HF (red), ET (blue), and WT (green) are compared on three different time scales. As expected, the HR shows a completely stable pattern. The frequency difference at the major third (E4) is 2.6 Hz for ET and 1 Hz for WT, which can be seen in the vibration patterns below.



You may wonder why we don't just use harmonic fractions as the tuning standard. Keep in mind that the above analysis is for the C major chord only. If we tune C4, E4, and G4 in perfect harmony, some of other chords in the C major key will be significantly off. Moreover, there are a total of 24 major and minor keys. Thus, tuning is a process of compromising. The equal temperament tuning does not favor any particular key, at the sacrifice of a certain degree of deviation from perfect harmonies.

A <u>seventh chord</u> is a chord consisting of a triad plus a note forming an interval of a seventh above the chord's root. Using the C chord as an example, the C7 chord consists of C, E, G, and Bb. The Cmaj7 chord consists of C, E, G, and B. The addition of the 7th node mades the chord sounding more unstable. Using the graphing calculator, the waveforms for C and C7 are shown below, based on equal temperament tuning.



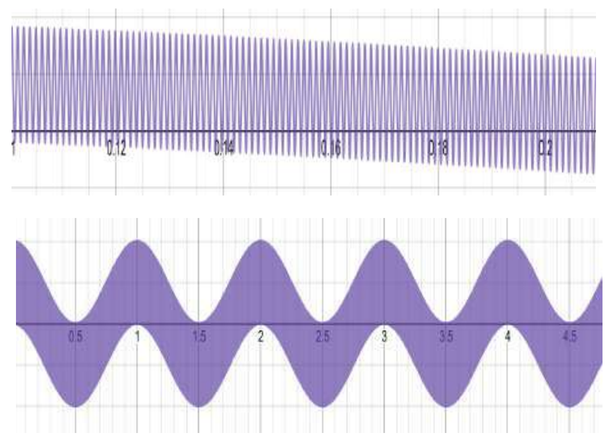
Beats

In acoustics, a beat is an interference pattern between two sounds of slightly different frequencies, perceived as a periodic variation in volume whose rate is the difference of the two frequencies [Wikipedia]. An interference can only be produced through a <u>nonlinear system</u>, not a <u>linear system</u>, as discussed below.

The output of a linear system is a linear combination of the inputs. The example in the previous section represents a linear system. The output $y(t)=\sin 2\pi f_{C4} t+\sin 2\pi f_{E4} t+\sin 2\pi f_{G4} t$ is a linear combination of the three inputs: $\sin 2\pi f_{C4} t$, $\sin 2\pi f_{E4} t$, and $\sin 2\pi f_{G4} t$. Thus, no new frequencies are generated. If the frequencies are not of exact harmonic fractions, some amplitude-modulated patterns can be observed. The amplitude does oscillates at the differential frequency of frequencies. However, the beat can only occur through a nonlinear system such as multiplying the two signals together. Using one of the trigonometry identities:

$$\sin\alpha \times \sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

Let $\alpha=2\pi 441 t$ and $\beta=2\pi 440 t$. The resulting waveform is shown on the right on two different time scales. Two new frequencies are generated: 1 Hz and 881 Hz. The lower frequency (1 Hz) is called the beat frequency. The beating can be used to tune a musical instrument, such as tuning two guitar strings to unison. When the pitches are close but not identical, the beat can be heard and used to guide the tuning. The 1 Hz difference between 441 Hz and 440 Hz is equivalent to 4 cents ($1200\log_2(441/440)$), which is not distinguishable by human ear in general. However, the beat frequency of 1 Hz can create a modulation on the sound volume perceived as a wobbling effect, which can be easily detected. When the two pitches are farther away, the wobbling is faster. When the two pitches are closer together, the wobbling is slower. The wobbling disappears when the two pitches are in perfect unison. A demonstration of this phenomenon can be seen and heard on YouTube entitled “Beats Demo: Tuning Forks” at <https://www.y-outube.com/watch?v=yia8spG8OmA>.



Conversion Between Frequencies and Cents

Ying Sun

Geometric series (frequencies)		**Algebraic series** (semitones or cents)
$\{f_0, f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12}\}$		$\{c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}\}$
$\{f_0,\ f_0 r,\ f_0 r^2,\ f_0 r^3,\ f_0 r^4,\ f_0 r^5,\ f_0 r^6,\ f_0 r^7,\ f_0 r^8,\ f_0 r^9,\ f_0 r^{10},\ f_0 r^{11},\ f_0 r^{12}\}$ $r = \sqrt[12]{2} = 2^{\frac{1}{12}} = 1.059463094$		$\{c_0,\ c_0+100,\ c_0+200,\ c_0+300,\ c_0+400,\ c_0+500,\ c_0+600,\ c_0+700,\ c_0+800, c_0+900,\ c_0+1000,\ c_0+1100,\ c_0+1200\}$
$c_2 - c_1 = 1200 \ \log_2 \frac{f_2}{f_1}$ =1200*LOG(F2/F1;2)	$\Rightarrow$	$c_2 - c_1$
$\frac{f_2}{f_1}$	$\Leftarrow$	$\frac{f_2}{f_1} = 2^{\frac{(c_2 - c_1)}{1200}}$ =2^((C2-C1)/1200)
Multiply, Divide		Add, Subtract

Arranged for guitar. Listen at <www.youtube.com/watch?v=MKyMKzGzXjE> and follow the chord progression.

Prelude in C (BWV 846)

From the Well-Tempered Clavier

Johann Sebastian Bach

6 =D

Arranged for solo guitar
Ying Sun 2019

Piano Score of Bach's BWV 846

Prelude and Fugue in C

From the Well Tempered Clavier

Johann Sebastian Bach


20
24
28
32
36
Fugue
40


43
46
50
54
57
60

Well-tempered vs. equal-tempered tunings <https://www.youtube.com/watch?v=6OxXE3GLgJk>

Prélude No. 22 in B♭ Minor

from “Das Wohltemperierte Klavier” Book I

BWV 867

Johann Sebastian Bach
(1685 - 1750)


13
15
17
20
22
I ---->
F
Db6+
Gm7
C7
F7
Db7
Bbm
Gbmaj7
Adim7
Bbm
Bbm