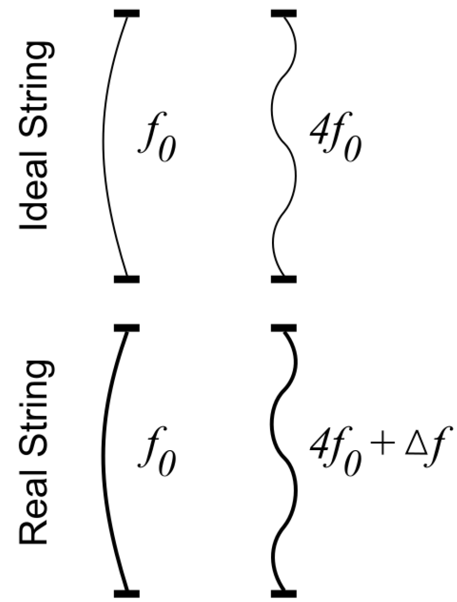
Inharmonicity

Inharmonicity is the degree to which the frequencies of the overtones (harmonics) depart from whole multiples of the fundamental frequency. The pitch of a note and its overtones define the timbre, which is the perceived sound quality of the musical note. Many percussion instruments, such as cymbals, tam-tams, and chimes, create complex and inharmonic sounds. However, in stringed instruments such as the piano, violin, and guitar, the overtones are close to whole number multiples of the fundamental frequency. Any departure from this ideal harmonic series is known as inharmonicity. In theory a perfectly flexible string would generate overtones that are in tune with the harmonic series. The less flexible the string is, the more inharmonicity it exhibits. A string is less flexible when it is shorter, thicker, stiffer, heavier, and lower in tension.



As real strings have some stiffness, the upper overtones are always slightly sharp, and progressively sharper. As shown by the example in the figure, the fundamental frequency of a string is f_0. For an ideal string, the 4th overtone should have the frequency of $4 \times f_0$. A real string is less flexible and has some thickness to it. The 4th overtone results on a slightly higher tension on the string, making the frequency slightly higher $4 f_0 + \Delta f$. An example of the harmonic frequencies for an A440 note is shown below:

Harmonics	1	2	3	4	5	6	7	8
Designated (Hz)	440	880	1320	1760	2200	2460	3080	3520
Measured (Hz)	440	881	1324	1770	2220	2493	3135	3603
Sharp by (Hz)	0	1	4	10	20	33	55	83
Sharp by (cents)	0	2	5	10	16	23	31	40

Mathematically, the frequency of the *n*th overtone is given by:

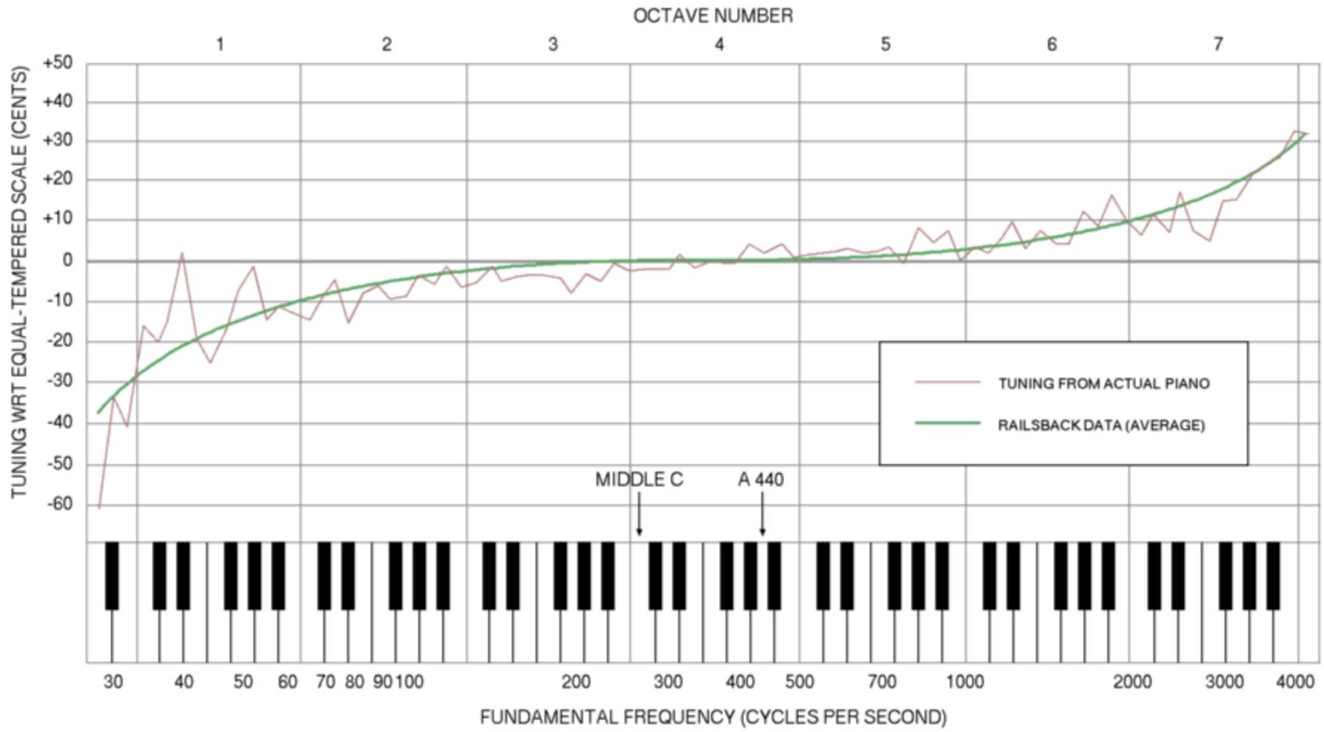
$$f_n = n\ f_0 \sqrt{1+\beta n^2}$$; The inharmonicity coefficient β is given by: $$\beta = \frac{\pi^3 Y D^4}{64 T L^2},$$

where Y is the Young's modulus (elasticity), D the diameter, T the tension, and L the length of the string. A larger β means a larger degree of inharmonicity and is associated with a string of higher rigidity, larger diameter, lower tension, and shorter length.

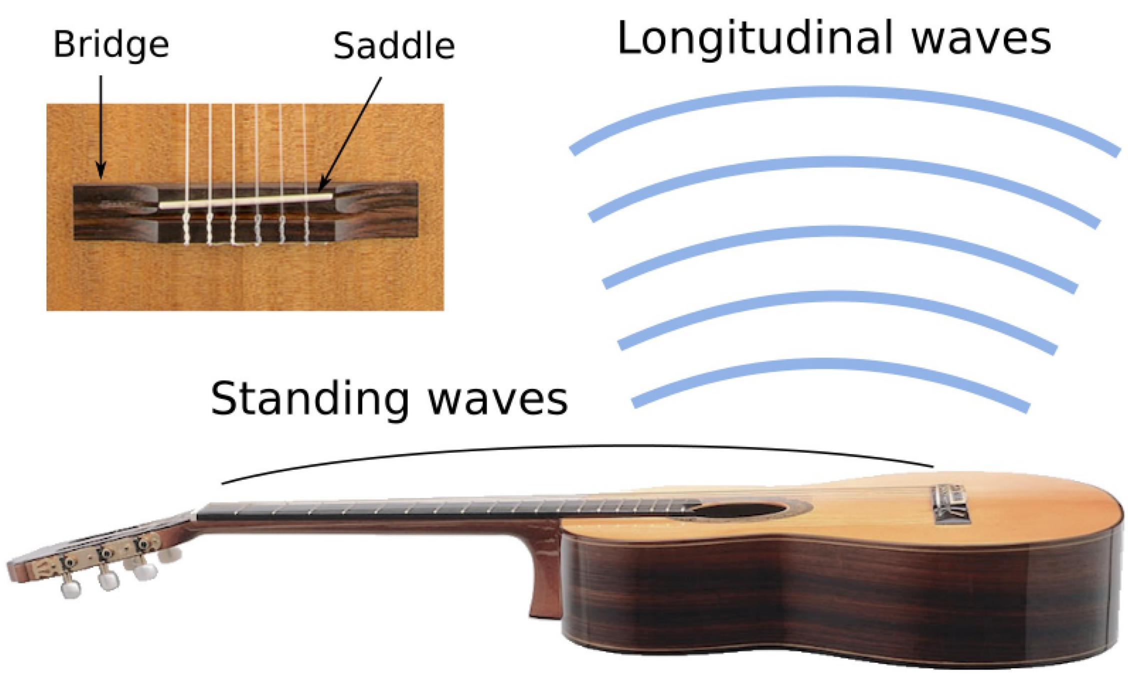
While the term “inharmonicity” seems to imply discord, it is in fact what gives string tones their warmth and character. The slight rise in their upper partials leads to a stretching of the octave that the human ear seems to expect or prefer and instruments tuned in this manner are often referred to as more *alive*.

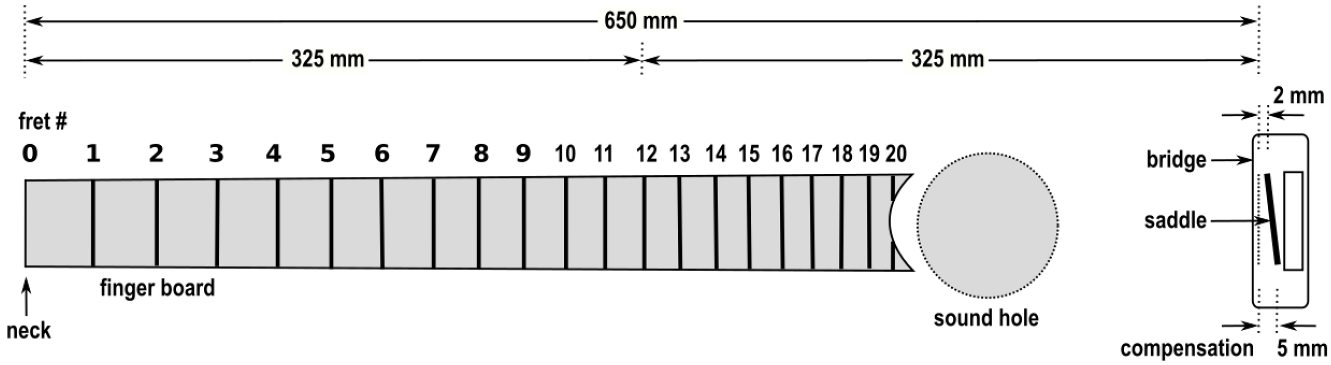
However, inharmonicity must be considered in tuning a traditional acoustic piano. Using the example in the above table, the 2nd harmonic of the A4 note is 881 Hz. If the A5 note is tuned to 880 Hz. Playing A4 and A5 together would result in a beating frequency of 1 Hz, which is perceived as a wobbling effect. Thus, a piano is usually not tuned to the designated frequencies of the individual notes; it needs to be tuned with a stretch. The higher registers are tuned sharp and the lower registers are tuned flat. This stretched tuning makes the piano sound more harmonious and pleasant overall.

The strings of the piano are short for the high registers and thick for the low registers. Thus, the phenomenon of inharmonicity is particularly prominent at the high and low registers. The amount of stretch tuning depends on the individual pianos and the personal preference of the piano tuner. The figure below shows an example of stretched tuning.



Guitar Acoustics and Saddle Compensation





The strings on the guitar produce standing waves, which are transferred to the sound board via the bridge. The vibrations of the sound board create longitudinal waves to project out. The sound board is made of a thin layer of a soft wood such as ceder or spruce. Wooden braces are glued to the under side of the sound board to provide structural supports. The side and back of the body of the guitar are made of a tone wood, which is a hard wood such as rosewood (for classical guitar). The neck is made of a wood that resists warping, such as mahogany. The finger board is made of a dense wood such as ebony.

Fret #	An from saddle (mm)	Bn from neck (mm)
0	650	0
1	614	36
2	579	71
3	547	103
4	516	134
5	487	163
6	460	190
7	434	216
8	409	241
9	386	264
10	365	285
11	344	306
12	325	325
13	307	343
14	290	360
15	273	377
16	258	392
17	243	407
18	230	420
19	217	433
20	205	445

The frets are usually made of white copper wires. The position of frets can be calculated by assuming that the frequency is inversely proportional to the length of the string. Refer to the figure above, let the distance from the *n*th fret to the saddle be A_n and assume equal temperament tuning:

$$A_n = \frac{L}{2^{n/12}}$$

, where *L* is the length of the string from neck to saddle. Let B_n be the distance from the *n*th fret to the neck (fret #0). We have

$$B_n = L - A_n \quad .$$

The typical string length is 650 mm. Table on the right shows the positions of the frets based on the 650 mm scale. The 12th fret is right in the middle, which is 325 mm from the neck and 325 mm from the saddle.

However, the saddle needs to be positioned with a proper compensation in order to make the *intonation* more accurate. This is because pressing the string onto the fret adds more tension to the string and causes the pitch to rise slightly. This problem is more prominent for the base strings because they are thicker. The figure above shows that the saddle is moved away from the neck by about 2 mm for the 1st (treble) string and 5 mm for the 6th (base) string.