1. Consider the following objective and constraint functions:

$$J(\mathbf{x}, \mathbf{u}) = \frac{1}{2}(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})$$
$$\mathbf{0} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} - \mathbf{c}$$

where $\mathbf{Q}$ is a symmetric, positive semi-definite matrix (not necessarily invertible), $\mathbf{R}$ is a symmetric positive definite matrix (invertible), and $\mathbf{A}$ is an invertible matrix. The vector $\mathbf{c}$ is a given vector. The objective of this problem is to find the optimal values of $\mathbf{x}$ and $\mathbf{u}$ that minimize J while satisfying the constraint equation. Note that $\mathbf{x}$ and $\mathbf{u}$ are constant vectors (i.e. not functions of time).

(a) Using a vector of Lagrange multipliers, $\mathbf{p}$, write an (unconstrained) objective function $J_a(\mathbf{x}, \mathbf{u}, \mathbf{p})$ that can be used to find the optimal solution $(\mathbf{x}^*, \mathbf{u}^*)$ of the original constrained optimization problem.

(b) Set derivatives of J_a with respect to $\mathbf{x}$, $\mathbf{u}$, and $\mathbf{p}$ to zero to get a set of equations characterizing the optimal solution.

(c) Solve the equations in the previous part to obtain expressions for $\mathbf{x}^*$ and $\mathbf{u}^*$.

(d) Evaluate your solution using the following numerical data. Verify that your solution satisfies the constraint equation and compute the numerical value of the objective function $J(\mathbf{x}^*, \mathbf{u}^*)$.

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 2.8720 & 1 & 0 \\ -2.7507 & 0 & 1 \\ 0.87868 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

2. Consider the following objective and constraint functions:

$$J(\mathbf{x}_1, \mathbf{x}_2, \mathbf{u}_0, \mathbf{u}_1) = \frac{1}{2}(\mathbf{x}_1^T\mathbf{Q}\mathbf{x}_1 + \mathbf{x}_2^T\mathbf{Q}\mathbf{x}_2 + \mathbf{u}_0^T\mathbf{R}\mathbf{u}_0 + \mathbf{u}_1^T\mathbf{R}\mathbf{u}_1)$$
$$\mathbf{0} = \mathbf{A}\mathbf{x}_0 + \mathbf{B}\mathbf{u}_0 - \mathbf{x}_1$$
$$\mathbf{0} = \mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1 - \mathbf{x}_2$$

where $\mathbf{Q}$ is a symmetric, positive semi-definite matrix (not necessarily invertible), $\mathbf{R}$ is a symmetric positive definite matrix (invertible), and $\mathbf{A}$ is an invertible matrix. The vector $\mathbf{x}_0$ is given vector. The objective of this problem is to find the optimal values of the constant vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{u}_0$ and $\mathbf{u}_1$ that minimize J while satisfying the two constraint equations.

(a) Using two vectors of Lagrange multiplier vectors, $\mathbf{p}_1$ and $\mathbf{p}_2$, write an (unconstrained) objective function J_a that can be used to find the optimal solution of the original constrained optimization problem.

(b) Set derivatives of J_a to zero to get a set of equations characterizing the optimal solution.

(c) Solve the equations in the previous part to obtain expressions for $\mathbf{x}_1^*, \mathbf{x}_2^*, \mathbf{u}_0^*$ and $\mathbf{u}_1^*$. Your expressions will be in terms of $\mathbf{x}_0$.

(d) Evaluate your solution using the numerical data given in Problem 1 (let $\mathbf{x}_0$ in this problem equal $\mathbf{c}$ from Problem 1). Verify that your solution satisfies the constraint equations and compute the numerical value of the objective function at the optimal solution.

3. Consider the following Hamiltonian matrix:

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T \\ -\mathbf{Q} & -\mathbf{A}^T \end{bmatrix}.$$

The purpose of this problem is to verify two important properties of the Hamiltonian matrix: (a) it is a symplectic matrix (that is, if λ is an eigenvalue of $\mathbf{M}$ then $-\lambda$ is also and eigenvalue; and (b) n of the eigenvalues of $\mathbf{M}$ are the closed-loop poles of the steady-state linear quadratic regulator system $(\mathbf{A}, \mathbf{B}, \mathbf{K})$ and the other n eigenvalues are the negatives of these poles. Recall that the LQR feedback gain matrix is $\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$.

Consider the following nonsingular matrix $\mathbf{T}$ and its inverse:

$$\mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{P} & \mathbf{I} \end{bmatrix}, \ \mathbf{T}^{-1} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{P} & \mathbf{I} \end{bmatrix},$$

where $\mathbf{P}$ is the solution to the following algebraic Riccati equation:

$$\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = \mathbf{0}.$$

Calculate the matrix $\mathbf{T}\mathbf{M}\mathbf{T}^{-1}$ and use the result to explain why properties (a) and (b) given above are true. [Hint: use the following Facts from Chapter 2 of *Digital Control: A State-Space Approach* by R.J. Vaccaro: 2.29, 2.31, 2.38, and 2.40.]

4. Given a plant model $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, consider the following cost function

$$J = \frac{1}{2}\int_0^{t_f} \Big[[\mathbf{r}(t) - \mathbf{C}\mathbf{x}(t)]^T \mathbf{Z} [\mathbf{r}(t) - \mathbf{C}\mathbf{x}(t)] + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t) \Big] \, dt.$$

Use Lagrange multipliers and derivatives to derive the optimal solution for this control problem, where $\mathbf{r}(t)$ is a given (vector-valued) reference input function.

(a) Follow the procedure given in class for the LQR problem. Note that the differential equation for the state and costate (Lagrange multiplier) vectors will *not* be homogeneous.

(b) Note that $\mathbf{r}^T(t)\mathbf{Z}\mathbf{C}\mathbf{x}(t) = \mathbf{x}^T(t)\mathbf{C}^T\mathbf{Z}^T\mathbf{r}(t)$ because the transpose of a scalar equals itself.

(c) The solution for

$$\begin{bmatrix} \mathbf{x}(t_f) \\ \mathbf{p}(t_f) \end{bmatrix}$$

contains an integral involving $\mathbf{r}(t)$. You do *not* have to evaluate this integral. Simply give it the following name:

$$\mathbf{v}(t) = \begin{bmatrix} \mathbf{v}_1(t) \\ \mathbf{v}_2(t) \end{bmatrix}.$$

Note that $\mathbf{p}(t_f) = \mathbf{0}$ because $\mathbf{H} = \mathbf{0}$ for this problem.

(d) You should end up with the following relationship:

$$\mathbf{p}(t) = \mathbf{P}(t)\mathbf{x}(t) + \mathbf{v}_2(t).$$

Differentiate this equation to obtain two differential equations, one for $\mathbf{P}(t)$ and the other for $\mathbf{v}_2(t)$.

(e) Write a final expression for the optimal input $\mathbf{u}(t)$ and give a block diagram of the optimal control system.

5. Consider the following plant

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \text{ with } \mathbf{R} = 1 \text{ and } \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}.$$

   (a) Find $\mathbf{P}$, the solution of the algebraic Riccati equation using the eigenvectors of the Hamiltonian matrix, and $\mathbf{K}$ the feedback gain vector. Find the poles of the resulting closed-loop regulator.

   (b) Write a program to solve the Riccati differential equation. Describe how you choose the sampling interval. How many seconds does it take for $\mathbf{P}(t)$ to converge to the matrix $\mathbf{P}$ found in part (a)?

6. Repeat (a) and (b) from the previous problem using the system given in Example 6.8 in the book.

7. Give an approximate formula for the amount of time it takes for $\mathbf{P}(t)$ to converge to $\mathbf{P}$ for a general LQR problem. [Hint: consider the relationship between eigenvalues and settling time.]