1. (Problem 3.3) Given a stationary, zero-mean random process $w(t)$ with correlation function

$$R_w(\tau) = \sigma_w^2 e^{-a|\tau|}$$

   (a) Obtain an expression for the spectral density of $w(t)$.

   (b) Obtain an expression for the correlation time of $w(t)$.

2. (Problem 3.4) Given the system

$$\dot{x}(t) = -2x(t) + 3w(t); \quad y(t) = 4x(t)$$

   where $w(t)$ is white noise with a spectral density of 16, and $x(0) = 0$, do the following:

   (a) Compute the variance of $x(t)$ using the integral expression for the state covariance matrix.

   (b) Compute the variance of $x(t)$ using the differential equation for the state covariance matrix.

3. (Problem 3.7) The plant described by the model

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -8 & -4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u, \quad y = [1 \quad 0 \quad 0]\,\mathbf{x}$$

   is being controlled by the following proportional controller

$$u(t) = -3[y(t) + v(t)]$$

   where $v(t)$ is measurement noise. The measurement noise is assumed to be white with a spectral density of 5. For the closed-loop system, find the steady-state covariance matrices of the state and output.

4. (Computer Exercise 3.1) A tachometer is used to measure the angular velocity, $\omega(t)$ in deg/sec, of an ac motor. The tachometer measurement, $\hat{\omega}(t)$, includes noise. Filtering may be used to reduce the effect of this noise and therefore generate improved angular velocity estimates. Filtering parameters can be adjusted to optimize the accuracy of the estimates. The variance of the estimation error, $e(t) = \omega(t) - \hat{\omega}(t)$, is used as a measure of estimator accuracy.

   A mathematical model for an ac motor is

$$\dot{\omega}(t) = \mathbf{A}\omega(t) + \mathbf{b}[u(t) + w(t)], \quad \mathbf{A} = -0.1, \mathbf{b} = 2.$$

   The ac voltage applied to the motor is the sum of a deterministic voltage $u(t)$ and a random voltage $w(t)$, both in volts. The random applied voltage is assumed to be white noise with the following spectral density:

$$S_w = 1 \frac{volts^2}{Hz}.$$

The noisy angular velocity measurement (output equation) is

$$y(t) = \mathbf{C}\omega(t) + v(t), \quad \mathbf{C} = 1.$$

The measurement noise $v(t)$ is assumed to be white noise with the following spectral density:

$$S_v = 100 \frac{deg^2}{sec^2 - Hz}$$

and to be uncorrelated with the plant noise.

An observer can be used to estimated angular velocity (the plant state variable). The observer is described by the following differential equation:

$$\dot{\hat{\omega}}(t) = (\mathbf{A} - \mathbf{GC})\hat{\omega}(t) + \mathbf{b}u(t) + \mathbf{G}y(t)$$

where $\mathbf{G}$ is the observer gain.

The estimation error variance can be computed from the combined plant/observer model. In what follows, the control input, $u(t)$, is equal to zero and the system is responding only to the noise. Do the following as a function of $\mathbf{G}$.

(a) Obtain a state-space model for the combined plant/observer system.

(b) Solve the Lyapunov equation for the steady-state state covariance matrix of the combined model and use this to find the estimation error variance.

(c) Plot the variance of the estimation error as a function of $\mathbf{G}$ and find the value of $\mathbf{G}$ that minimizes this variance.

(d) Plot the observer pole location $(\mathbf{A} - \mathbf{GC})$ as a function of $\mathbf{G}$. What observer pole gives the smallest estimation error variance?