ELE/MCE 504 **Homework #6** **Spring '09**

1. The results of Problem 6(b) of Homework 5 are as follows:

$$\dot{\mathbf{\Sigma}}_e(t) = (\mathbf{A} - \mathbf{G}(t)\mathbf{C})\mathbf{\Sigma}_e(t) + \mathbf{\Sigma}_e(t)(\mathbf{A} - \mathbf{G}(t)\mathbf{C})^T + [\,\mathbf{B}_w \quad -\mathbf{G}(t)\,] \begin{bmatrix} \mathbf{S}_w & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_v \end{bmatrix} \begin{bmatrix} \mathbf{B}_w^T \\ -\mathbf{G}^T(t) \end{bmatrix}.$$

   Substitute in the expression

$$\mathbf{G}(t) = \mathbf{\Sigma}_e(t)\mathbf{C}^T\mathbf{S}_v^{-1}$$

   and put the results in the form of a matrix Riccati differential equation for $\mathbf{\Sigma}_e(t)$.

2. (Computer Exercise 3.1) A tachometer is used to measure the angular velocity, $\omega(t)$ in deg/sec, of an ac motor. The tachometer measurement, $\hat{\omega}(t)$, includes noise. The variance of the estimation error, $e(t) = \omega(t) - \hat{\omega}(t)$, is used as a measure of estimator accuracy.

   A mathematical model for an ac motor is

$$\dot{\omega}(t) = \mathbf{A}\omega(t) + \mathbf{b}[u(t) + w(t)], \quad \mathbf{A} = -0.1, \mathbf{b} = 2.$$

   The ac voltage applied to the motor is the sum of a deterministic voltage $u(t)$ and a random voltage $w(t)$, both in volts. The random applied voltage is assumed to be white noise with the following spectral density:

$$S_w = 1\frac{volts^2}{Hz}.$$

   The noisy angular velocity measurement (output equation) is

$$y(t) = \mathbf{C}\omega(t) + v(t), \quad \mathbf{C} = 1.$$

   The measurement noise $v(t)$ is assumed to be white noise with the following spectral density:

$$S_v = 100\frac{deg^2}{sec^2 - Hz}$$

   and to be uncorrelated with the plant noise.

   A steady-state Kalman filter can be used to estimated angular velocity (the plant state variable). The Kalman filter is described by the following differential equation:

$$\dot{\hat{\omega}}(t) = (\mathbf{A} - \mathbf{G}\mathbf{C})\hat{\omega}(t) + \mathbf{b}u(t) + \mathbf{G}y(t)$$

   where $\mathbf{G}$ is the Kalman gain. Recall that the Kalman filter minimizes the estimation error variance.

   (a) Use Matlab to calculate the value of $\mathbf{G}$ for this problem.

   (b) Assume that the value of $\mathbf{S}_v$ is changed to 10. Recompute the Kalman gain.

3. (Problem 7.5 (c), (d)) Given the following plant

$$\dot{x}(t) = 2x(t) + 2u(t) + w(t)$$
$$m(t) = 3x(t) + v(t)$$

   where $w(t)$ and $v(t)$ are white noise signals, uncorrelated with each other, with spectral densities, $S_w = 1$ and $S_v = 5$, respectively.

(a) Calculate the steady-state Kalman gain using the eigenvector decomposition of the Hamiltonian matrix approach, and verify your result using the Matlab `care` function.

(b) What is the pole of the steady-state Kalman filter?

(c) How does the pole change if the value of S_v is changed to 50?

4. Consider the following MIMO plant model for the hover mode pitch and yaw dynamics of a vertical-takeoff-and-landing remotely piloted vehicle. The first state variable is pitch angle and the third state variable is yaw angle, both in degrees.

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1.6 \\ 0 & 0 & 0 & -1 \\ 0 & 1.8 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 0 \\ -11 & 0 \\ 0 & 0 \\ 0 & -12 \end{bmatrix} \mathbf{u}(t) + \begin{bmatrix} 0 & 0 \\ -11 & 0 \\ 0 & 0 \\ 0 & -12 \end{bmatrix} \mathbf{w}(t)$$

$$\mathbf{m}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}(t) + \mathbf{v}(t)$$

where $\mathbf{w}(t)$ and $\mathbf{v}(t)$ are white noise signals, uncorrelated with each other, with spectral densities, $\mathbf{S}_w = 10\mathbf{I}$ (10 times an identity matrix) and $\mathbf{S}_v = 10^{-4}\mathbf{I}$, respectively. Let the weighting matrices in the steady-state stochastic LQR cost function be

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.02 \end{bmatrix}.$$

(a) Compute the optimal LQR feedback gain matrix $\mathbf{K}$.

(b) Compute closed-loop LQR poles and the stability robustness norm for the LQR regulator.

(c) Compute the steady-state state covariance matrix of $\mathbf{x}(t)$ for the closed-loop LQR system. Note that the diagonal entries of this matrix are the variances of the regulated state variables. In particular, the (1,1) element is the variance of the pitch angle and the (3,3) element is the variance of the yaw angle. Take the square roots of these numbers to get the standard deviation in degrees.

(d) Compute the Kalman gain matrix $\mathbf{G}$ for a Kalman filter for this plant.

(e) Compute Kalman filter poles and the stability robustness norm for the LQG regulator.

(f) How much "faster" are the Kalman filter poles than the LQR poles?

(g) Compute the steady-state state covariance matrix for $[\,\mathbf{x}^T(t) \quad \hat{\mathbf{x}}^T(t)\,]^T$. How well does the LQG regulator work compared with the LQR regulator in terms of the variances of the plant state variables x_1 and x_3?

(h) Repeat (d)-(g) after changing the observation noise spectral density to $\mathbf{S}_v = 0.01\mathbf{I}$.