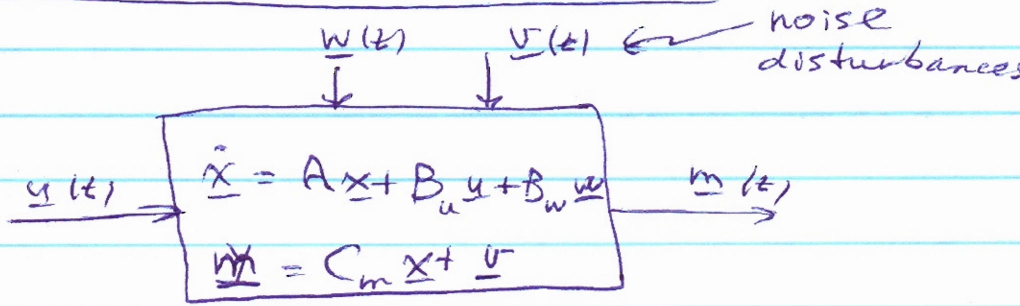
Set up Kalman filter Problem

Given



How can we estimate the plant state variables? Simple approach: "ignore the disturbances and use an observer"

observer equation

$$\dot{\hat{\underline{x}}}(t) = (A - GC)\hat{\underline{x}}(t) + B_u\underline{u}(t) + G\underline{m}(t)$$

KF approach: generate MMSE optimal estimate of $\dot{\underline{x}}(t)$. from knowledge of $\underline{u}(t)$ and $\underline{m}(t)$

Assumptions:

A1: $E[\underline{w}(t)\,\underline{w}^T(t+\tau)] = S_w\,\delta(\tau)$ (white noise)

A2: $E\{\underline{v}(t)\,\underline{v}^T(t+\tau)\} = S_v\,\delta(\tau)$

A3: $E\{\underline{v}(t)\,\underline{w}^T(t+\tau)\} = 0 \;\forall \tau$ (v, $\underline{w}$ uncorrelated)

A4: $E[\underline{x}(0)\,\underline{w}^T(t)] = 0 \quad \forall t \geq 0$

A5: $E[\underline{x}(t)\,\underline{v}^T(t+\tau)] = 0 \quad \forall \tau$

A6: Thus $E[\underline{x}(t)\,\underline{w}^T(t+\tau)] = 0 \quad \forall \tau > 0$ because $\underline{x}(t) = e^{At}\underline{x}(0) + \int_0^t e^{A(t-\tau)} B\underline{w}(\tau)\,d\tau$

(not an assumption)

Generate MMSE estimate of $\underline{x}(t)$, call it $\hat{\underline{x}}(t)$, such that

$$J = E[(\underline{x}(t) - \hat{\underline{x}}(t))^T(\underline{x}(t) - \hat{\underline{x}}(t)) \text{ is minimized.}$$

Approach: Construct a linear, time varying estimate of $\dot{\underline{x}}(t)$, call it $\hat{\dot{\underline{x}}}(t)$

(E) $\hat{\dot{\underline{x}}}(t) = F(t)\,\hat{\underline{x}}(t) + G(t)\,\underline{m}(t) + H(t)\,u(t)$

(estimator equation) [This is what we will end up with] (need equations to specify F, G, H)

Start Here

Consider two data sets: (D1): $\underline{m}(\tau)$, $0 \leq \tau \leq t - \varepsilon$

(D2): $\underline{m}(t)$ [a single vector]

Let $\hat{\dot{\underline{x}}}_\varepsilon(t)$ be the best estimate of $\dot{\underline{x}}(t)$ given data set D1.

$\hat{\dot{\underline{x}}}_\varepsilon(t)$ Characterized by orthogonality

(O1): $E[(\dot{\underline{x}}(t) - \hat{\dot{\underline{x}}}_\varepsilon(t))\,\underline{m}^T(\tau)] = 0, \quad 0 \leq \tau \leq t - \varepsilon$

Let $\underline{\hat{m}}_\varepsilon(t)$ be the best estimate of $\underline{m}(t)$ based on data set D1.

$\underline{\hat{m}}_\varepsilon(t)$ is characterized by orthogonality

O2 $E[(\underline{m}(t) - \underline{\hat{m}}_\varepsilon(t))\, \underline{m}^T(\tau)] = 0, \quad 0 \le \tau \le t-\varepsilon$

Then, from the two-measurement formula defined for random vectors,

unknown gain matrix

$$\underline{\hat{\dot{x}}}(t) = \underline{\hat{\dot{x}}}_\varepsilon(t) + G(t)\left(\underline{m}(t) - \underline{\hat{m}}_\varepsilon(t)\right)$$

Guess that $\boxed{\underline{\hat{\dot{x}}}_\varepsilon(t) = A\underline{\hat{x}}_\varepsilon(t) + B_u \underline{u}(t)}$ G1

where $\underline{\hat{x}}_\varepsilon(t)$ is the best estimate of $\underline{x}(t)$ given data set D1.

$\underline{\hat{x}}_\varepsilon(t)$ is characterized by orthogonality:

O3 $E[(\underline{x}(t) - \underline{\hat{x}}_\varepsilon(t)]\, \underline{m}^T(\tau)] = 0 \quad 0 \le \tau \le t-\varepsilon$

Show that G1 is correct by showing that G1 and O3 satisfy O1:

HW { Plant state eq

G1 → O1 yields $E[(A\underline{x}(t) + B_u\underline{u}(t) + B_w\underline{w}(t) - A\underline{x}_\varepsilon(t) - B_u\underline{u}(t))\, \underline{m}^T(\tau)] = 0$

using O3 $\quad 0 \le \tau \le t-\varepsilon$

and assumptions A5 and A6

Guess that:

(G2) $\hat{\underline{m}}_\varepsilon(t) = C\,\hat{\underline{x}}_\varepsilon(t)$

Show that (G2) and (O3) satisfy (O2):

HW: O2: $E\left\{ \left(\underbrace{C\underline{x}(t) + \underline{v}(t)}_{\text{plant measurement equation}} - C\hat{\underline{x}}_\varepsilon(t) \right) \underline{m}^T(\tau) \right\} = 0$ $\quad 0 \le \tau \le t-\varepsilon$

using (O3) and (A5)

So estimator equation (E) for $\dot{\underline{x}}(t)$ becomes

(E1) $\dot{\hat{\underline{x}}}(t) = A\hat{\underline{x}}_\varepsilon(t) + B_u \underline{u}(t) + G(t)\left[\underline{m}(t) - C\hat{\underline{x}}_\varepsilon(t)\right]$

take $\lim_{\varepsilon \to 0}$ so $\underline{x}_\varepsilon(t)$ becomes $\hat{\underline{x}}(t)$

Use Fubini's theorem (interchange expectation and differentiation so that $\dot{\hat{\underline{x}}}(t) \to \dot{\hat{\underline{x}}}(t)$.

Then E1 becomes

(E2) $\dot{\hat{\underline{x}}}(t) = \underbrace{[A - G(t)C]}_{F(t)}\hat{\underline{x}}(t) + \underbrace{B_u}_{H(t)}\underline{u}(t) + G(t)\underline{m}(t)$

Last thing to find is $G(t)$.

Remark: the derivation is unchanged if the plant model is time varying:

$A(t),\ B_u(t),\ C(t)$

(E2) shows that the Kalman filter is an observer with a time-varying gain matrix, $G(t)$, called the Kalman gain.

The only unknown in E2 is the Kalman gain matrix, $G(t)$.

Using the orthogonality principal,

(OT) $$E\left[(\underline{x}(t) - \hat{\underline{x}}(t))\, \underline{m}^T(\tau) \right] = 0 \qquad 0 \le \tau \le t$$

It can be shown ~~that~~:

$$\boxed{G(t) = \Sigma_e(t)\, C^T S_v^{-1}}$$ (KGE) Kalman gain equation

where the estimation error covariance matrix is

$$\Sigma_e(t) = E\left\{ (\underline{x}(t) - \hat{\underline{x}}(t))(\underline{x}(t) - \hat{\underline{x}}(t))^T \right\} = E\left[(x - \hat{x})\, x^T \right]$$

If $\underline{e}(t) \triangleq \underline{x}(t) - \hat{\underline{x}}(t)$, then

$$\dot{\underline{e}}(t) = \dot{\underline{x}}(t) - \dot{\hat{\underline{x}}}(t)$$

$$\dot{\underline{e}}(t) = (A - G(t)C)\, \underline{e}(t) + \begin{bmatrix} B_w & -G(t) \end{bmatrix} \begin{bmatrix} \underline{w}(t) \\ \underline{v}(t) \end{bmatrix}$$

From Chapter 3 we know that the state covariance matrix ~~$\Sigma_x(t)$~~ for $\dot{\underline{x}} = A\underline{x} + B\underline{w}$ satisfies:

$$\dot{\Sigma}_x(t) = A\Sigma_x(t) + \Sigma_x(t)A^T + B S_w B^T$$

Thus

$$\dot{\Sigma}_e(t) = (A - G(t)C)\, \Sigma_e(t) + \Sigma_e(t)\, (A - G(t)C)^T$$

$$+ \begin{bmatrix} B_w & -G(t) \end{bmatrix} \begin{bmatrix} S_w & 0 \\ 0 & S_v \end{bmatrix} \begin{bmatrix} B_w^T \\ -G^T(t) \end{bmatrix}$$

[Show for HW]

ARE's are the same

$$\dot{\Sigma}_e(t) = \Sigma_e(t) A^T + A \Sigma_e(t) + B_w S_w B_w^T - \Sigma_e(t) C^T S_v^{-1} C \Sigma_e(t)$$

$\Sigma_e(0)$ = specified
integrate forwards

a matrix Riccati differential equation!

Recall LQR: $\begin{cases} \dot{x} = Ax + Bu \\ J = \int (x^T(t) Q x(t) + u^T R u)\, dt \end{cases}$

$$K(t) = R^{-1} B^T P(t) \quad \text{where}$$

$$\dot{P}(t) = -P(t) A - A^T P(t) - Q + P(t) B R^{-1} B^T P(t)$$

$P(t_f) = H$. Integrate backwards. Better to work w/ $-\dot{P}(t)$

These problems are identical with the following correspondences:

	LQR	KF	
	A	A^T	
	B	C^T	
	$K(t)$	$G^T(t)$	$G^T(t) = S_v^{-1} C \Sigma_e(t)$
	Q	$B_w S_w B_w^T$	$G(t) = \Sigma_e(t) C^T S_v^{-1}$
	R	S_v	
$K(t) = R^{-1} B^T P(t)$	$P(t)$ ($P(t_f)$ known)	$\Sigma_e(t)$, $\Sigma_e(0)$ known; $G^T(t)$	
$-\dot{P}(t)$	integrate Riccati backwards	integrate Riccati $\Sigma_e(t)$ forwards	
steady state	$P = U_{21} U_{11}^{-1}$	$\Sigma = U_{22} U_{21}^{-1}$	in book p 261. But he uses ...

Steady-State Kalman filter.

Set $\dot{\Sigma}_e(t) = 0$ and solve ARE.

(minus sign is not important; multiply through by -1).

$$\mathcal{H} = \begin{bmatrix} A^T & -C^T S_v^{-1} C \\ -B_w S_w B_w^T & -A \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \left[\begin{array}{c|c} -\Lambda & 0 \\ \hline 0 & \Lambda \end{array}\right] U^{-1}$$

note: the book uses $-\mathcal{H}$ for the KF Hamiltonian.

$$\Sigma_e = U_{21} U_{11}^{-1}$$

$$G = \Sigma_e C^T S_v^{-1}$$

KF: $\dot{\hat{\underline{x}}}(t) = (A - GC)\hat{\underline{x}}(t) + B_u \underline{u}(t) + G \underline{m}(t)$

where are the steady-state KF poles?

[We should choose the pole locations for pole-placement observer design.]