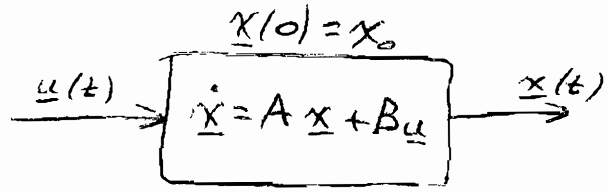
The Linear Quadratic Regulator

Given



Find the optimal input function, call it $\underline{u}^*(t)$, which minimizes the following cost function

$$\underset{(\underline{x}(t),\, \underline{u}(t))}{J} = \frac{1}{2}\underline{x}^T(t_f)\, H\, \underline{x}(t_f) + \frac{1}{2}\int_0^{t_f}\left[\underline{x}^T(t)\, Q\, \underline{x}(t) + \underline{u}^T(t)\, R\, \underline{u}(t)\right] dt$$

subject to $\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t)$, $t \geq 0$

H, Q, and R are symmetric, positive-definite matrices. We want to make $\underline{x}(t)$ small (drive to zero) without using excessive control effort.

Multivariable Constrained Optimization

Consider a scalar-valued function of a vector-valued variable, $J(\underline{x})$. Suppose we want to minimize $J(\underline{x})$ subject to the constraint $c(\underline{x}) = 0$.

write down all derivative formulas (see pg. 28)

$$\nabla_{\underline{x}} J = \frac{\partial J}{\partial \underline{x}} = \left[\frac{\partial J}{\partial x_1} \cdots \frac{\partial J}{\partial x_n}\right]$$

gradient is a row vector

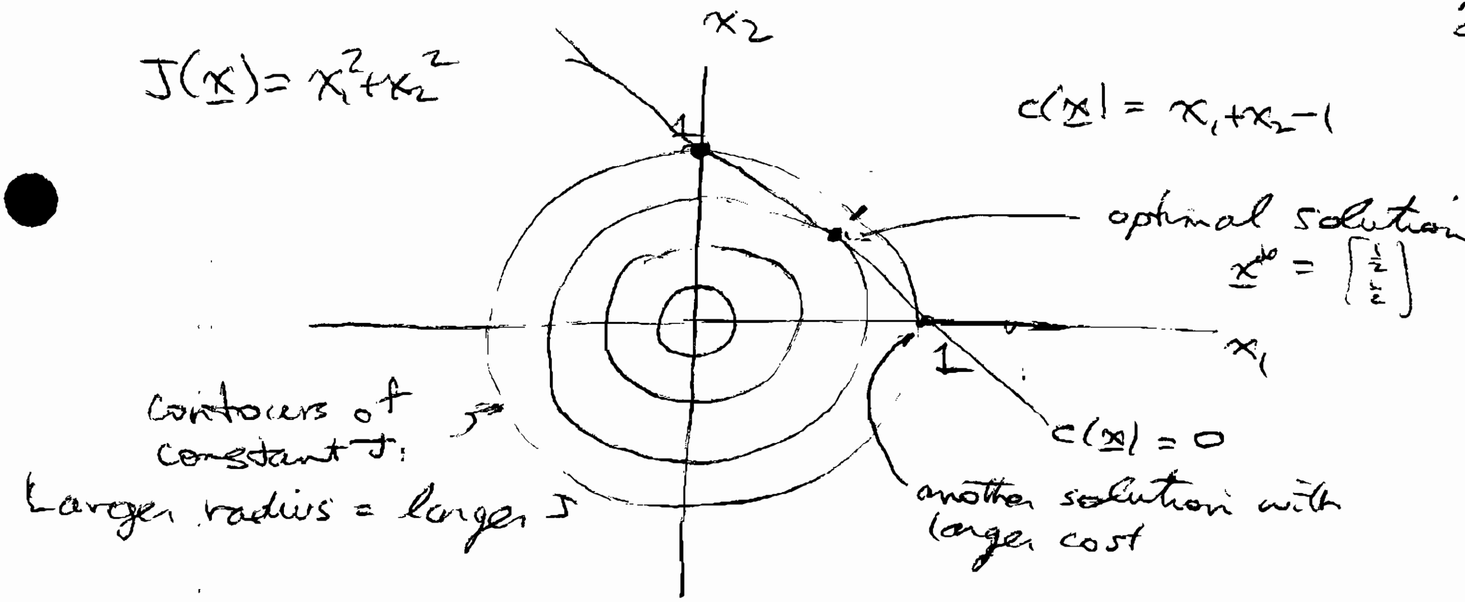
Example: $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $J(\underline{x}) = x_1^2 + x_2^2$

$c(\underline{x}) = x_1 + x_2 - 1$

We can get the solution to this optimization problem by inspection!

$J(\underline{x}) = x_1^2 + x_2^2$

$c(\underline{x}) = x_1 + x_2 - 1$



How can we find the optimal solution analytically?

Form augmented cost function:

$$J_a(\underline{x}, p) = J(\underline{x}) + \underbrace{p}_{\text{Lagrange multiplier}}\, c(\underline{x})$$

$$= x_1^2 + x_2^2 + p(x_1 + x_2 - 1)$$

Set $\left.\dfrac{\partial J_a}{\partial \underline{x}}\right|_{\underline{x}=\underline{x}^*} = \underline{0}^T$ ① and $\left.\dfrac{\partial J_a}{\partial p}\right|_{\underline{x}=\underline{x}^*} = 0$ ②

① $\left[(2x_1^* + p) \quad (2x_2^* + p)\right] = \left[0 \quad 0\right]$

② $x_1^* + x_2^* - 1 = 0$

① → $x_1^* = -\dfrac{p}{2}, \quad x_2^* = -\dfrac{p}{2}$ ③

③ → ② $-\dfrac{p}{2} - \dfrac{p}{2} - 1 = 0 \Rightarrow \boxed{p = -1}$ ④

④ → ③ $\boxed{x_1^* = \dfrac{1}{2}, \quad x_2^* = \dfrac{1}{2}}$

How to handle more than one constraint:

e.g. min $J(\underline{x})$ subject to $c_1(\underline{x}) = 0$
and $c_2(\underline{x}) = 0$

Form $J_a(\underline{x}) = J(\underline{x}) + \underline{p}^T \underline{c}(\underline{x})$

where $\underline{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$; $\underline{c}(\underline{x}) = \begin{bmatrix} c_1(\underline{x}) \\ c_2(\underline{x}) \end{bmatrix}$

Set $\left.\dfrac{\partial J_a}{\partial \underline{x}}\right|_{\underline{x}=\underline{x}^*} = \underline{0}^T$ and $\underbrace{\left.\dfrac{\partial J_a}{\partial \underline{p}}\right|_{\underline{x}=\underline{x}^*} = \underline{0}^T}_{\text{equivalent to } \underline{c}(\underline{x}) = \underline{0}}$

Extend to four constraints:
$c_1(\underline{x}) = 0$, $c_2(\underline{x}) = 0$, $d_1(\underline{x}) = 0$ $d_2(\underline{x}) = 0$

One way: $\underline{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$, $\underline{c}(\underline{x}) = \begin{bmatrix} c_1(\underline{x}) \\ c_2(\underline{x}) \\ d_1(\underline{x}) \\ d_2(\underline{x}) \end{bmatrix}$, $\underline{p}^T \underline{c}(\underline{x})$ (5)

Another way: Let $\underline{p}_1 = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$, $\underline{p}_2 = \begin{bmatrix} p_3 \\ p_4 \end{bmatrix}$

$\underline{c}(\underline{x}) = \begin{bmatrix} c_1(\underline{x}) \\ c_2(\underline{x}) \end{bmatrix}$, $\underline{d}(\underline{x}) = \begin{bmatrix} d_1(\underline{x}) \\ d_2(\underline{x}) \end{bmatrix}$

use $\underline{p}_1^T \underline{c}(\underline{x}) + \underline{p}_2^T \underline{d}(\underline{x})$, which is same as (5)
[add up the different constraints]

$= p_1 c_1(\underline{x}) + p_2 c_2(\underline{x}) + p_3 d_1(\underline{x}) + p_4 d_2(\underline{x})$

add up constraint functions each with their

Lagrange multiplier

Extend to an infinite number of constraints: addition becomes integral.

LQR

$$J(\underline{x}(t), \underline{u}(t)) = \frac{1}{2}\underline{x}^T(t_f) H \underline{x}(t_f) + \frac{1}{2}\int_0^{t_f}\left(\underline{x}^T(t) Q \underline{x}(t) + \underline{u}^T(t) R \underline{u}(t)\right)dt$$

subject to $\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t)$ } one constraint for each value of t

Form augmented cost function:

$$J_a(\underline{x}(t), \underline{u}(t), \underline{p}(t)) = J(\underline{x}, \underline{u}) + \underbrace{\int_0^{t_f} \underline{p}^T(t)\left[A\underline{x}(t) + B\underline{u}(t) - \dot{\underline{x}}(t)\right]dt}_{\text{add up the constraints at each } t.}$$

For future reference, recall integration by parts:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du \qquad \Big\} \quad \dot{\underline{x}}\,dt = d\underline{x} = dv, \quad \dot{\underline{p}}^T dt = d\underline{p}^T = du, \quad u = \underline{p}^T \; v = \underline{x}$$

$$\int_0^{t_f} \overbrace{\underline{p}^T(t)}^{u}\, \overbrace{\dot{\underline{x}}(t)\,dt}^{dv} = \underline{p}^T(t_f)\underline{x}(t_f) - \underline{p}^T(0)\underline{x}(0) - \int_0^{t_f} \dot{\underline{p}}^T(t)\,\underline{x}(t)\,dt$$

General formulas: $\frac{\partial}{\partial \underline{u}}(R\underline{u}) = R$

If $M = M^T$, $\frac{\partial}{\partial \underline{u}}(\underline{u}^T M \underline{u}) = 2\underline{u}^T M$

Variables are $\{\underset{\text{not } \dot{x}(t)}{\underbrace{\underline{x}(t)}}, \underline{u}(t), \underline{p}(t)\}$ $0 \le t \le t_f$

Rewrite augmented cost function using integration by parts formula:

$$J_a = \tfrac{1}{2}\, \underline{x}^T(t_f)\, H\, \underline{x}(t_f) + \tfrac{1}{2}\int_0^{t_f} \left(\underline{x}^T(t)\, Q\, \underline{x}(t) + \underline{u}^T(t) R \underline{u}(t)\right) dt$$

constraint term
$$\left\{ \begin{aligned} &+ \int_0^{t_f} \underline{p}^T(t) \left[A\, \underline{x}(t) + B\underline{u}(t)\right] dt \\ &- \underline{p}^T(t_f)\, \underline{x}(t_f) + \underline{p}^T(0)\, \underline{x}(0) + \int_0^{t_f} \dot{\underline{p}}^T(t)\, \underline{x}(t)\, dt \end{aligned} \right.$$

Set derivatives equal to zero:

① $$\frac{\partial J_a}{\partial \underline{x}(t_f)} = \underline{x}^T(t_f)\, H - \underline{p}^T(t_f) = \underline{0}^T$$

② $0 < t < t_f$ $$\frac{\partial J_a}{\partial \underline{x}(t)} = \underline{x}^T(t)\, Q + \underline{p}^T(t)\, A + \dot{\underline{p}}^T(t) = \underline{0}^T$$

③ $0 \le t \le t_f$ $$\frac{\partial J_a}{\partial \underline{u}(t)} = \underline{u}^T(t)\, R + \underline{p}^T(t)\, B = \underline{0}^T$$

④ $0 \le t \le t_f$ $$\frac{\partial J_a}{\partial \underline{p}(t)} = \underline{x}^T(t)\, A^T + \underline{u}^T(t)\, B^T - \dot{\underline{x}}^T(t) = \underline{0}^T$$

(from original expression for J_a, before the integration by parts)

From ③

$$\underline{u}^*(t) = -R^{-1} B^T \underline{p}^*(t) \qquad ⑤$$

Combine ② & ④ & ⑤

$$⑥ \quad \begin{bmatrix} \dot{\underline{x}}^*(t) \\ \dot{\underline{p}}^*(t) \end{bmatrix} = \overbrace{\left[\begin{array}{c|c} A & -BR^{-1}B^T \\ \hline -Q & -A^T \end{array}\right]}^{\text{call this } \mathcal{H}} \begin{bmatrix} \underline{x}^*(t) \\ \underline{p}^*(t) \end{bmatrix} \qquad 0 < t \leq t_f$$

Mixed boundary conditions
$$\begin{cases} \underline{x}^*(0) = \underline{x}_0 \quad (\text{given}) \\ ⑦ \ \underline{p}^*(t_f) = H\,\underline{x}^*(t_f) \qquad \text{from } ① \end{cases}$$

⑥ is a homogeneous state-space model. Given the solution at any initial time t, we can compute the solution at a future time t_f:

recall: $\boxed{\begin{array}{l} \dot{\underline{x}} = Ax \\ \underline{x}(t) = e^{A(t-t_0)}\,\underline{x}(t_0) \end{array}}$

$$\begin{bmatrix} \underline{x}^*(t_f) \\ \underline{p}^*(t_f) \end{bmatrix} = \underbrace{e^{\mathcal{H}(t_f - t)}}_{\text{call this matrix}} \begin{bmatrix} \underline{x}^*(t) \\ \underline{p}^*(t) \end{bmatrix}$$

all are a function of time

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \qquad (2n \times 2n)$$

Write out the previous equation:

use ⑦
$$\begin{cases} \underline{x}^*(t_f) = M_{11}\,\underline{x}^*(t) + M_{12}\,\underline{p}^*(t) \\ \underline{p}^*(t_f) = M_{21}\,\underline{x}^*(t) + M_{22}\,\underline{p}^*(t) \end{cases}$$

$$M_{21}\,\underline{x}^*(t) + M_{22}\,\underline{p}^*(t) = HM_{11}\,\underline{x}^*(t) + HM_{12}\,\underline{p}^*(t)$$

$$(M_{22} - H M_{12})\, \underline{p}^*(t) = (H M_{11} - M_{21})\, \underline{x}^*(t)$$

or

$$\underline{p}^*(t) = \underbrace{(M_{22} - H M_{12})^{-1} (H M_{11} - M_{21})}_{\substack{\text{this is an } n\times n \\ \text{matrix-valued function} \\ \text{of time. Call it } \underbrace{P(t)}_{n\times n}}} \underline{x}^*(t) \qquad 0 < t \le t_f$$

$$\underline{p}^*(t) = P(t)\, \underline{x}^*(t) \qquad (8)$$

Plug (8) into (5): $\boxed{\underline{u}^*(t) = -R^{-1} B^T P(t)\, \underline{x}^*(t)}$

The optimal solution is given by (time-varying) <u>state feedback.</u> Gain matrix is

$$K(t) = R^{-1} B^T P(t)$$

(but $P(t)$ is currently unknown)

Find $P(t)$ by solving the (matrix) Riccati (differential) equation

Differentiate (8): $\dot{\underline{p}}^*(t) = \dot{P}(t)\, \underline{x}^*(t) + P(t)\, \dot{\underline{x}}^*(t)$

Use (6)

$$-Q\, \underline{x}^*(t) - A^T \underline{p}^*(t) = \dot{P}(t)\, \underline{x}^*(t) + P(t)\left[A \underline{x}^*(t) - B R^{-1} B^T \underline{p}^*(t)\right]$$

Substitute in (8)

$$\left[\dot{P}(t) = -P(t) A - A^T P(t) - Q + P(t) B R^{-1} B\, P(t)\right] \underline{x}^*(t)$$

The previous equation must be true for any $\underline{x}^*(t)$ (recall $\underline{x}_0$ is arbitrary), thus the matrix equation must be satisfied:

$$\dot{P}(t) = -P(t)A - A^T P(t) - Q + P(t) B R^{-1} B^T P(t) \qquad (9)$$

$$0 < t < t_f$$

From (8) & (7)

$$P(t_f) = H \qquad (10)$$

(9) is a (matrix valued) nonlinear differential equation in $P(t)$ with a given final (not initial) condition. It can be solved numerically by integrating backward in time from t_f to 0.

Choose a sampling interval, T, to be a small negative number. Then

$$t_k = t_f + kT$$

$$(t_0 = t_f)$$

$$\dot{P}(t_k) \approx \frac{P(t_k) - P(t_{k+1})}{t_k - t_{k+1}}$$

$$= \frac{P(t_k) - P(t_{k+1})}{-T} = \frac{P(t_{k+1}) - P(t_k)}{T}$$

Initialize with $P(t_0) = H$

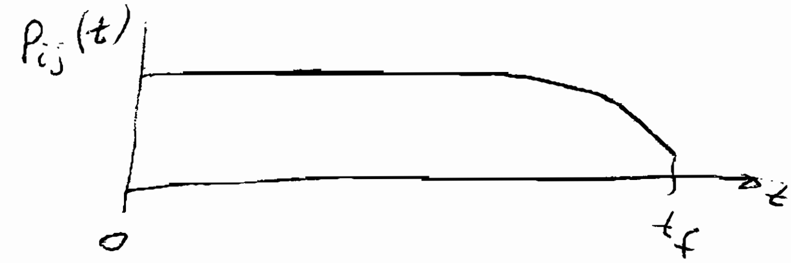
$$P(t_{k+1}) = P(t_k) + T\left[-P(t_k)A - A^T P(t_k) - Q + P(t_k) B R^{-1} B^T P(t_k)\right]$$

Continue until $t_{k+1} = 0$

As t_f gets large, the elements of $P(t)$

$$P(t) = [\, P_{ij}(t) \,]$$

have the following behavior:



as $t_f \to \infty$, $P(t)$ becomes a constant matrix, P. Thus $\underline{p}(t) = P\,\underline{x}(t)$

The Steady-State Linear Quadratic Regulator

$t_f \to \infty$, $\quad H = 0$, $\quad \underline{p}(t_f) = \underline{0} \quad$ $\underline{0}$ as $t_f \to \infty$

$(\underline{p}(t_f) = P(t)\,\underline{x}(t_f))$

Riccati differential equation becomes the algebraic Riccati equation ($\dot{P}(t) = 0$ because $P(t) = P$, a constant matrix)

$$PA + A^T P - PBR^{-1}B^T P + Q = \mathbf{0} \qquad \text{(ARE)}$$

In Matlab >> P = care (A, B, Q, R)

Let us derive the solution to the CARE. Recall

$$\begin{bmatrix} \dot{\underline{x}}(t) \\ \dot{\underline{p}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}}_{\mathcal{H}} \begin{bmatrix} \underline{x}(t) \\ \underline{p}(t) \end{bmatrix}$$

, $\underline{x}(0) = \underline{x}_0$

$\lim_{t \to \infty} \underline{p}(t) = \underline{0}$

and

$\lim_{t \to \infty} \underline{x}(t) = \underline{0}$

$\mathcal{H}$ is a matrix of real numbers.
Thus, if λ is a complex eigenvalue of $\mathcal{H}$, then λ^* is also an eigenvalue of $\mathcal{H}$.

You will show for homework that, if λ is an eigenvalue of $\mathcal{H}$, then $-\lambda$ is also an eigenvalue of $\mathcal{H}$. Thus, $\mathcal{H}$ is a <u>symplectic</u> matrix.
You will also show for HW that n eigenvalues of $\mathcal{H}$ are in LHP. The other n (their negatives) are in the RHP.

Let $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ contain the eigenvalues of $\mathcal{H}$ in the RHP.

Write eigendecomposition of $\mathcal{H}$ as

$$\underset{2n \times 2n}{\mathcal{H}} = \Big[\underbrace{\mathcal{U}_1}_{n} \Big| \underbrace{\mathcal{U}_2}_{n} \Big] \left[\begin{array}{c|c} -\Lambda & 0 \\ \hline 0 & \Lambda \end{array} \right] \Big[\mathcal{U}_1 \; \mathcal{U}_2 \Big]^{-1}$$

$$= \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} -\Lambda & 0 \\ 0 & \Lambda \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$$

partition $\mathcal{U}$ into $n \times n$ blocks

partition V into $n \times n$ blocks

Then $\begin{bmatrix} \underline{x}(t) \\ p(t) \end{bmatrix} = e^{\mathcal{H}t} \begin{bmatrix} \underline{x}_0 \\ p_0 \end{bmatrix}$

known

unknown

or

$$\begin{bmatrix} \underline{x}(t) \\ \underline{p}(t) \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \left[\begin{array}{c|c} e^{-\Lambda t} & 0 \\ \hline 0 & e^{\Lambda t} \end{array}\right] \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \underline{x}_0 \\ \underline{p}_0 \end{bmatrix}$$

$$\underline{x}(t) = U_{11}\, e^{-\Lambda t} \underbrace{(V_{11}\,\underline{x}_0 + V_{12}\,\underline{p}_0)}_{\text{same, call it } \underline{v}} + U_{12}\, \overset{\lim_{t\to\infty} \to \infty}{e^{\Lambda t}} \underbrace{(V_{21}\,\underline{x}_0 + V_{22}\,\underline{p}_0)}_{\text{must be } \underline{0}}$$

$$\underline{p}(t) = U_{21}\, e^{-\Lambda t} (V_{11}\,\underline{x}_0 + V_{12}\,\underline{p}_0) + U_{22}\, e^{\Lambda t} \overset{\underline{0}}{\cancel{(V_{21}\,\underline{x}_0 + V_{22}\,\underline{p}_0)}}$$

Recall $\underline{p}(t) = P\,\underline{x}(t)$

or

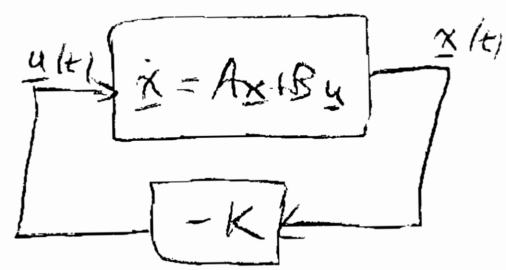
$$\boxed{U_{21}}\, e^{-\Lambda t}\, \underline{v} = \boxed{P\,U_{11}}\, e^{-\Lambda t}\, \underline{v} \quad \text{for arbitrary } \underline{v}.$$

equate

$$\boxed{P = U_{21}\, U_{11}^{-1}} \quad \text{solution to CARE}$$

Stability Robustness of LQR (steady state)

See next page for picture of Nyquist system



where
$\mathbf{K} = R^{-1} B^T P$

Recall for a state-feedback regulator, $\delta_{max} = \| \underbrace{(A-BK),\ B,\ -K,\ 0}_{\text{state-space model of } N_{z_d w_d}(s)} \|_\infty$ ← system ∞ norm