MMSE Estimation [Chapter 7]

Given a random vector measurement, $\underline{m}$, estimate the value of an unknown random vector, $\underline{x}$, using a linear estimator: $\underset{(n\times 1)}{\hat{\underline{x}}} = \underset{(n\times n)}{F}\,\underset{(n\times 1)}{\underline{m}}$ [Find F.]

Orthogonality principal: optimal estimate makes $\underline{x} - \hat{\underline{x}}$ orthogonal to $\underline{m}$.

$$E\left\{(\underline{x}-\hat{\underline{x}})\,\underline{m}^T\right\} = \underset{n\times n}{0}$$ orthogonal means no correlation

$$\underbrace{E\left[x\,\underline{m}^T\right]}_{C_{xm}} = F\,\underbrace{E\left[m\,m^T\right]}_{C_{mm}}$$

$$\boxed{\begin{array}{l} F = \underbrace{C_{xm}}_{n\times n}\,\underbrace{C_{mm}^{-1}}_{n\times n} \\ \hat{x} = Fm \end{array}}$$

minimizes $E\left[(\underline{x}-\hat{\underline{x}})^T(\underline{x}-\hat{\underline{x}})\right]$

Suppose we want to estimate $\underline{x}$ given two measurements, $\underline{m}_1$ and $\underline{m}_2$.

Brute Force Approach

$$\hat{x} = F_1\underline{m}_1 + F_2\underline{m}_2 = \underbrace{\begin{bmatrix} F_1 & F_2 \end{bmatrix}}_{\substack{F \\ n\times 2n}} \begin{bmatrix} \underline{m}_1 \\ \underline{m}_2 \end{bmatrix}$$ call it $\underset{2n\times 1}{\underline{m}}$

Then $[F_1 \;\; F_2] = \underbrace{C_{xm}}_{n\times 2n}\,\underbrace{C_{mm}^{-1}}_{2n\times 2n}$

and

$$E\left((\underline{x}-\hat{\underline{x}})\,[\underline{m}_1^T \;\; \underline{m}_2^T]\right) = 0$$ requires inverting a bigger matrix.

Recursive Idea

First estimate $\underline{m}_2$ from $\underline{m}_1$:

(recall $\underline{m}_2$ is known)

$$\hat{\underline{m}}_2 = G\,\underline{m}_1 \quad \text{where } G = C_{\underline{m}_2\underline{m}_1} C_{m_1 m_1}^{-1}$$

$$\text{and} \quad E\left[(\underline{m}_2 - \hat{\underline{m}}_2)\,\underline{m}_1^T\right] = 0$$

Then we can write $\underline{m}_2$ as

$$\underline{m}_2 = \underbrace{\hat{\underline{m}}_2}_{\text{predictable from } \underline{m}_1} + \underbrace{\underline{e}_2}_{\underline{m}_2 - \hat{\underline{m}}_2}$$

$\underline{e}_2$ contains information not found in $\underline{m}_1$.

where $E[\underline{e}_2 \underline{m}_1^T] = 0$ or $C_{e_2 m_1} = 0$

Idea: estimate $\underline{x}$ by summing individual estimates from $\underline{m}_1$ and $\underline{e}_2$:

Two measurement formula

$$\boxed{\hat{x} = F_1 \underline{m}_1 + F_2 \underline{e}_2, \quad \left\{ \begin{array}{l} F_1 = C_{x m_1} C_{m_1 m_1}^{-1} \\ F_2 = C_{x e_2} C_{e_2 e_2}^{-1} \end{array} \right.}$$

invert two $n \times n$ matrices

Is this estimate optimal? [Does it satisfy orthogonality principle?]

$$E\left[(\underline{x} - \hat{x})\,[\underline{m}_1^T \;\; \underline{m}_2^T]\right]$$

$$= E\left[(\underline{x} - F_1 \underline{m}_1 - F_2 \underline{e}_2)\,[\underline{m}_1^T \;\; \underline{m}_2^T]\right]$$