A property of the maximum singular value of a matrix is:

$$\bar{\sigma}(M_1 \cdot M_2) \le \bar{\sigma}(M_1) \cdot \bar{\sigma}(M_2)$$

[The ~~the~~ matrix norm $\|M\|_2 = \bar{\sigma}(M)$ and a property of any p-norm is $\|M_1 M_2\|_2 \le \|M_1\|_2 \|M_2\|_2$]

For all values of $\omega$,

$$\bar{\sigma}\big(G_1(j\omega)\, G_2(j\omega)\big) \le \bar{\sigma}\big(G_1(j\omega)\big)\, \bar{\sigma}\big(G_2(j\omega)\big)$$

Take the supremum over $\omega$ of both sides to get

(4.41) in book

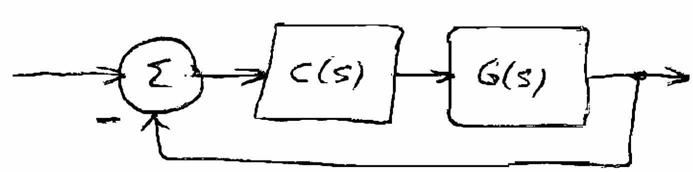
$$\boxed{\|G_1(s)\, G_2(s)\|_\infty \le \|G_1(s)\|_\infty \, \|G_2(s)\|_\infty}$$

## Stability Robustness of SISO Control Systems

[5.2 to 5.4]

Classical Control: given $G(s)$, calculate $C(s)$ so that the following system is stable:

(I)




Suppose the plant model is uncertain. With $C(s)$ staying the same, how much can $G(s)$ change before the control system goes unstable.

## Classical perturbation models: $G(s) = g \cdot \underbrace{G_o(s)}_{\text{nominal model}}$

- $g$ is a multiplicative perturbation.
- $g = 1 \Rightarrow G(s) =$ nominal system
- $g \cdot G_o(s) = G_o(s) \cdot g$ so input and output perturbations are the same.

**Gain Perturbation:** $\underbrace{g_{min}}_{<1} < g < \underbrace{g_{max}}_{>1}$
($g$ is a real number)

Control system remains stable when an unmodeled gain of $g$ multiplies the nominal plant model. How can we find $g_{min}$ and $g_{max}$?

**Phase Perturbation:** $g = e^{-j\phi\frac{\pi}{180}}$

Stability maintained for $-\phi_{max} < \phi < \phi_{max}$

$\phi$ represents unmodeled phase lag in degrees

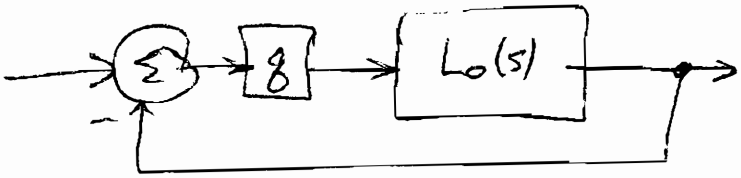
How can we find $\phi_{max}$?

**Nyquist Analysis**

Let $L(s) = G(s)C(s)$, $\quad L_o(s) = G_o(s)C(s)$

$$\begin{aligned} L(s) &= G(s)C(s) \\ &= g\,G_o(s)C(s) \\ &= g\,L_o(s) \end{aligned}$$

System (I) can be rewritten as




(II)

The CLTF for this model is

$$\frac{g L_o(s)}{1 + g L_o(s)}$$

A given complex number $s_1$ is a pole of the CL system iff: $1 + g L_o(s_1) = 0$, or

$$(*) \quad \boxed{L(s_1) = -\frac{1}{g}}$$

If this equality is satisfied by any number $s_1$ in RHP, the CL system is unstable.

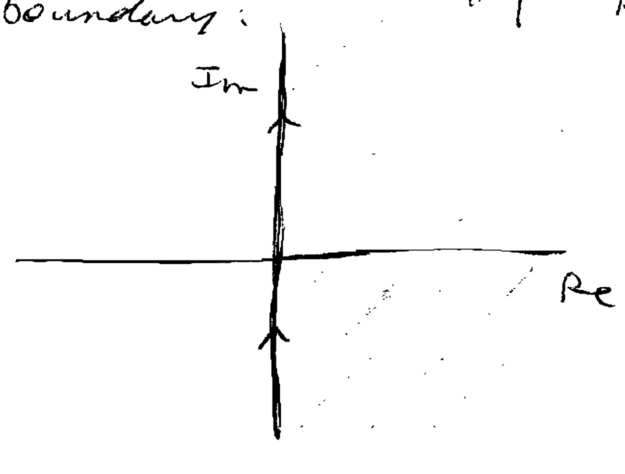
From (graduate level) complex theory

(1) An oriented boundary in the complex plane defines a set of numbers with "shading to the right."

(2) An oriented boundary in a complex plane, when mapped through a rational function, results in another oriented boundary, usually drawn on a second complex plane.

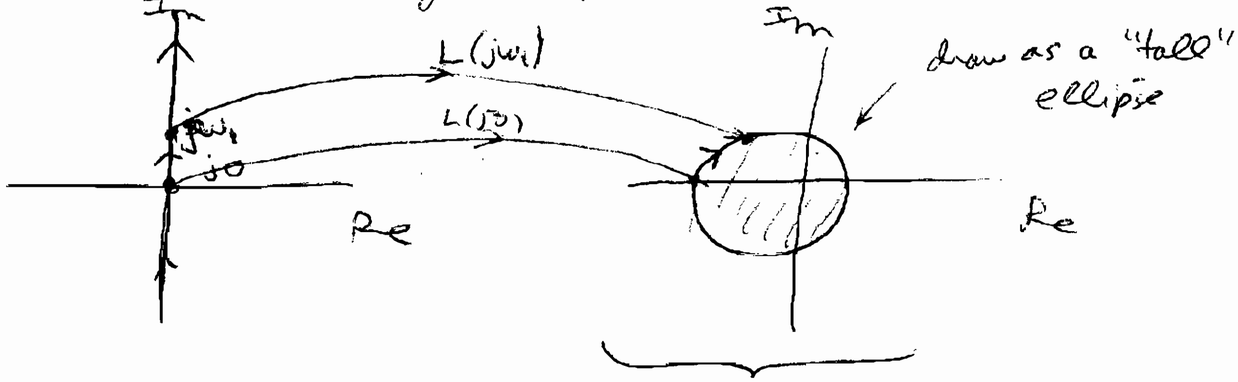
(3) Every point in the first shaded region maps to some point in the second shaded region

Consider the following oriented boundary:

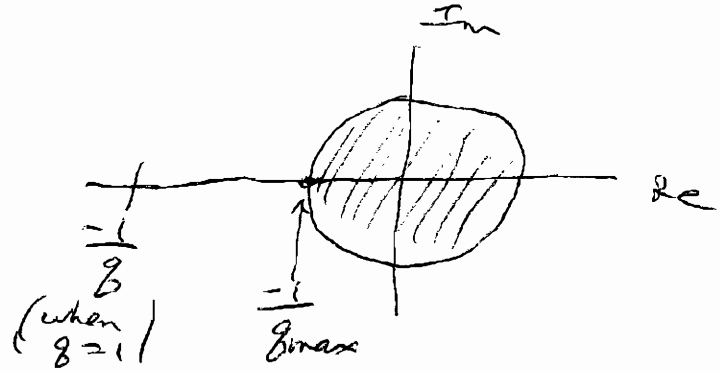



This boundary defines a set of points consisting of all numbers in RHP.

Take every number on the boundary, evaluate $L_0(s)$ at that number, and plot the resulting complex number:




Recall (*)



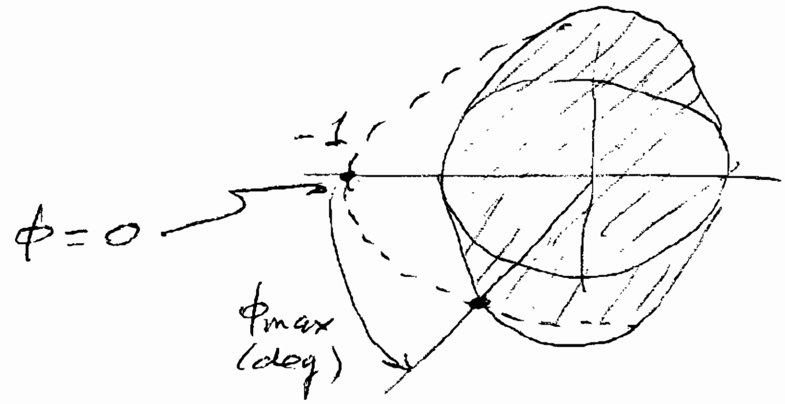

If $-\frac{1}{g}$ is <u>not</u> in shaded region, then $L(s) \neq -\frac{1}{g}$ for any $s$ in RHP, $\therefore$ system II is stable. For this picture, system II is stable for

$$\underset{g_{min}}{0} \le g \le g_{max}$$

$$UGM = 20\log_{10}(g_{max})\ dB$$
$$LGM = 20\log_{10}(g_{min})\ dB$$

From the picture (plot) we can also determine phase margin:

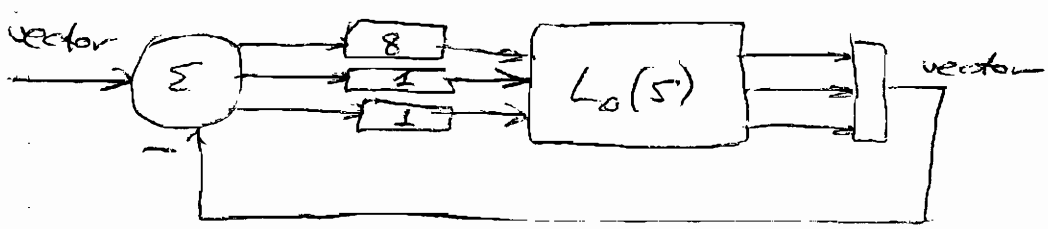
$$g = e^{-j\phi\pi/180}$$
$$1/g = e^{j\phi\pi/180}$$
$$-1/g = e^{j(\pi + \phi\pi/180)}$$




System (II) is stable for $-\phi_{max} < \phi < \phi_{max}$

How does a computer find $g_{max}$, $g_{min}$, $\phi_{max}$?

What about a MIMO system?
Easiest way to extend the classical approach is as follows:

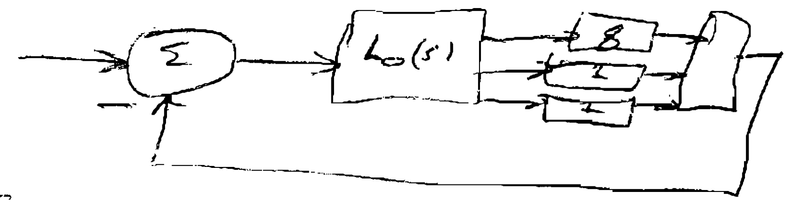



Put $g$ (gain or phase perturbation) on one input at a time. Calculate UGM, LGM, PM for each input channel using SISO approach.

## Drawbacks

($\Delta$1) Does not allow for simultaneous perturbations on all input channels (realistic).

($\Delta$2) Putting perturbations on the output channels will yield different results




## Remarks

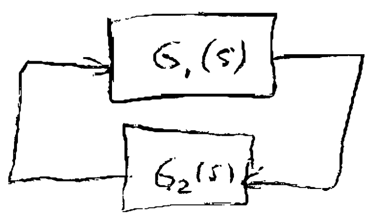
($\Delta$2) is an unavoidable fact of MIMO systems.

($\Delta$1) can be overcome by a modern approach to stability robustness (~ late 1970s)

## Unstructured Uncertainty Models (MIMO)

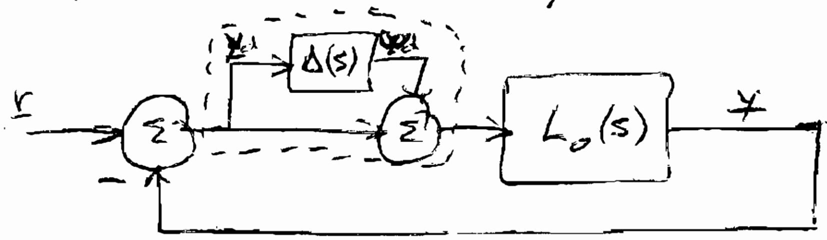
(later we will look at what this approach tells us for SISO systems)

Basic theorem for the feedback interconnection of two multivariable systems:




is stable iff $\|G_1(s)\,G_2(s)\|_\infty < 1$

Input multiplicative plant uncertainty model: cascade plant with MIMO TF $I+\Delta(s)$




- all lines represent vector signals
- nominal control ~~signal~~ system given by $\Delta(s) = 0$

Stability Robustness: how "big" can $\Delta(s)$ be before system goes unstable?

Find the TF from $\underline{w_d}$ to $y_d$ (with $\underline{r}$ set equal to zero)

$$y = L_o(s)\left[\underline{w_d} \cancel{+} - y\right]$$

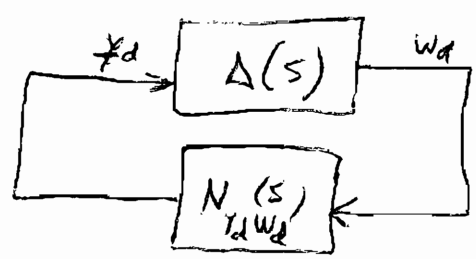
$$y_d = -y$$

$$(I+L_o(s))\,y = L_o(s)\,\underline{w_d}$$

$$y = (I+L_o(s))^{-1}L_o(s)\,\underline{w_d}$$

$$y_d = \underbrace{\boxed{-(I+L_o(s))^{-1}L_o(s)}}_{\text{call this } N_{y_d w_d}(s)}\,\underline{w_d}$$

Then III can be redrawn as



Condition for stability is (from basic theorem)

$$\| \Delta(s) \, N_{Y_d W_d}(s) \|_\infty < 1$$

Use (4.41)

$$\| \Delta(s) \, N_{Y_d W_d}(s) \|_\infty \le \boxed{\| \Delta(s) \|_\infty \, \| N_{Y_d W_d}(s) \|_\infty < 1}$$

Require this condition

Result: System III is stable iff

$$\| \Delta(s) \|_\infty < \frac{1}{\| N_{Y_d W_d}(s) \|_\infty}$$

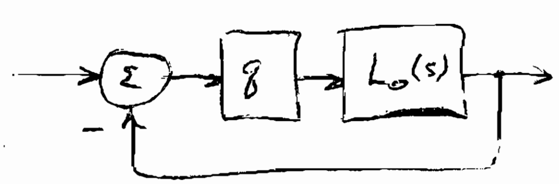
unknown, unstructured perturbation model, including possible gain and phase variations on all input channels.

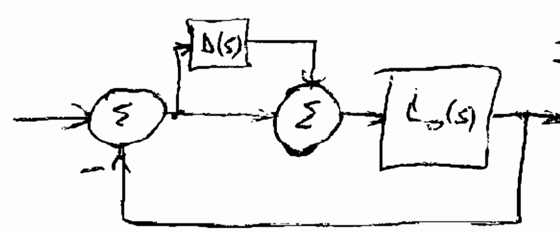
Call this number $\delta_{max}$

We will call $\delta_{max}$ the robustness norm. The bigger $\delta_{max}$ is, the larger the model uncertainties before the control system goes unstable.

<u>Relationship Between Classical & Norm Stability Robustness for SISO systems</u>

II




III

Equate the perturbations:

$$g = 1 + \Delta(s)$$

Suppose $\|\Delta(s)\|_\infty < \delta_{max}$

If $\Delta(s) = K$, a real #, then $|K| < \delta_{max}$, or $-\delta_{max} < K < \delta_{max}$

$$g_{max} = 1 + \delta_{max}$$
$$g_{min} = 1 - \delta_{max}$$

Recall $UGM = 20 \log_{10} g_{max}$
$LGM = 20 \log_{10} g_{min}$

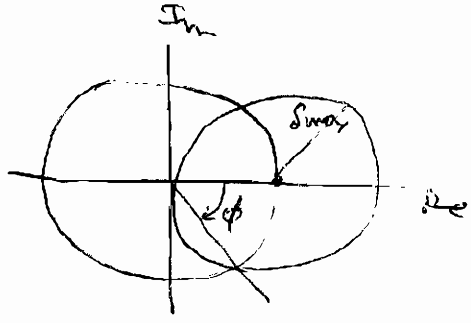
$3dB \Rightarrow g_{max} = \sqrt{2}$ (1.4)
$-3dB \Rightarrow g_{min} = \frac{1}{\sqrt{2}}$ (.707)

To get $g_{max} = \sqrt{2}$, we need $\delta_{max} = \sqrt{2} - 1 \approx 0.4$

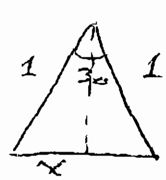
If $\delta_{max} > 1$, $g_{min} < 0$ and $LGM = -\infty\,dB$

How about phase margin? Let $\Delta(s)$ be a complex number $c$ with magnitude $< \delta_{max}$

$$g = e^{-j\phi} = \underset{\text{center}}{1} + \underset{\text{disc of radius } \delta_{max}}{c}$$




Notes:
(1) If $\delta_{max} > 2$ there is no phase margin
(2) To get $PM = 30°$,

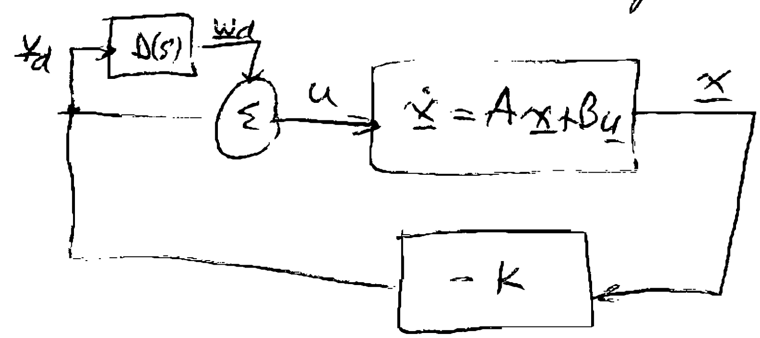



$x = \sin 15°$

$\delta_{max} = 2x = .5176$

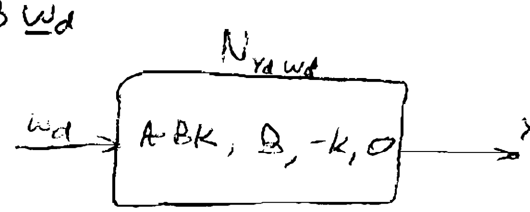
So rule of thumb, we want $\boxed{\delta_{max} \geq \frac{1}{2}}$

Example: Find the robustness norm for a state-feedback regulator.




Find state-space model for $N_{y_d w_d}$ :

$$\dot{\underline{x}} = A\underline{x} + B(\underline{w}_d - BK\underline{x})$$
$$= (A - BK)\,\underline{x} + B\,\underline{w}_d$$
$$y_d = -k\,\underline{x}$$

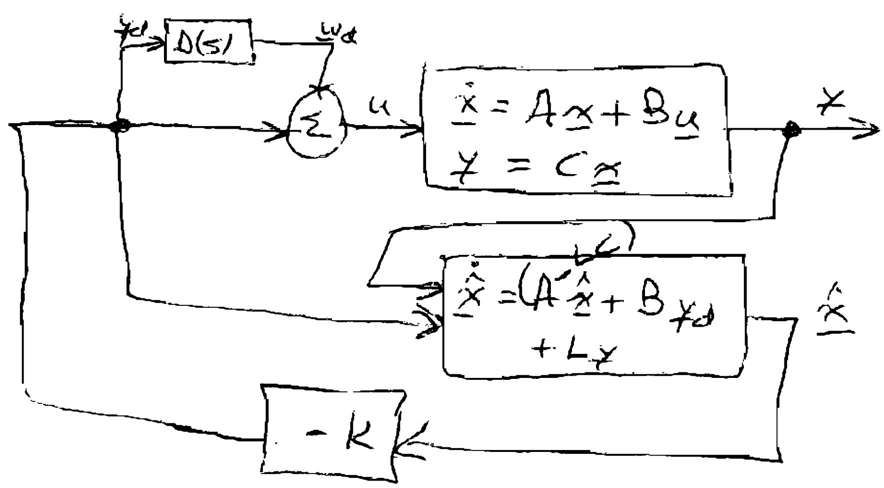



In Matlab:

```
>> sys = ss ( A-B*k, B, -k, zeros(p,p))
>> f = norm (sys, inf)
```

$= \| N_{y_d w_d}(s) \|_\infty$

Robustness norm for observer-based regulator:




observer looks at the signal that is <u>sent</u> to the plant