SHANNON SECOND THEOREM: IF $R < C$, WHERE R, C ARE IN BITS / CHANNEL USE, THEN AS $n \to \infty$, WE CAN TRANSMIT INFORMATION WITHOUT ERROR.

CALLED NOISY CHANNEL CODING THEOREM.

DOESN'T TELL US HOW TO DO THE CODING 

WHAT IS CHANNEL CAPACITY?

NEED TO FIRST DEFINE MORE ENTROPY CONCEPTS.

FOR (X, Y) JOINT ENTROPY IS

$$H(X, Y) = -\sum_{x,y} p(x,y) \log_2 p(x,y)$$

(X, Y) ARE JOINTLY DIST. DISCRETE RANDOM VARIABLES WITH PMF $p(x,y)$

NOT f NOW

UNCERTAINTY ABOUT (X, Y).

A CONDITIONAL UNCERTAINTY IS

$$H(X \mid Y = y) = -\sum_{x} p(x|y) \log_2 p(x|y)$$

$$= E_{X|Y}[\log_2 p(X|y)]$$

THE CONDITIONAL ENTROPY DEFINED AS

$$H(X|Y) = E_Y\left[H(X|Y=y)\right]$$

$$= \sum_y \underbrace{\left[-\sum_x p(x|y) \log_2 p(x|y)\right]}_{H(X|Y=y)} p(y)$$

$$= -\sum_x \sum_y p(x,y) \log_2 p(x|y)$$

THIS IS AVERAGE UNCERTAINTY ABOUT X ONCE Y IS OBSERVED.

<u>EXAMPLE</u>: MIGHT EXPECT

$$\underbrace{\text{UNCERT.}}_{\text{INFO.}}(X,Y) = \underbrace{\text{UNCERT}}_{\text{INFO.}}(Y) + \underbrace{\text{UNCERT}}_{\text{INFO.}}(X|Y)$$

$$H(Y) + H(X|Y) = -\sum_y p(y) \log_2 p(y) - \sum_x \sum_y p(x,y) \log_2 p(x|y)$$

$$= -\sum_y p(y) \log_2 p(y) - \sum_x \sum_y p(x,y) \log_2 \frac{p(x,y)}{p(y)}$$

$$= -\sum_y p(y) \log_2 p(y) + \sum_x \sum_y p(x,y) \log_2 p(y) - \sum_x \sum_y p(x,y) \log_2 p(x,y) = H(X,Y)$$

SEE ALSO EX. 12,16.

MUTUAL INFORMATION

$H(X)$ = UNCERTAINTY ABOUT X

$H(X|Y)$ = UNCERTAINTY ABOUT X ONCE Y IS OBSERVED

SHOULD HAVE $H(X|Y) \le H(X)$ WHY?

TO PROVE THIS LET $I(X;Y) = H(X) - H(X|Y)$

$$I(X;Y) = -\sum_x \sum_y p(x,y) \log_2 p(x) + \sum_x \sum_y p(x,y) \log_2 p(x|y)$$

$$= \sum_x \sum_y p(x,y) \log_2 \frac{p(x|y)}{p(x)}$$

$$= \sum_x \sum_y p(x,y) \log_2 \frac{p(x,y)}{p(x)\,p(y)} \qquad (*)$$

$$-I(X;Y) = \sum_x \sum_y p(x,y) \log_2 \frac{p(x)\,p(y)}{p(x,y)}$$

$$\le \sum_x \sum_y p(x,y) \left[\frac{p(x)\,p(y)}{p(x,y)} - 1 \right]$$

$$= \sum_x \sum_y p(x)\,p(y) - \sum_x \sum_y p(x,y) = 0$$

$I(x;y)$ IS CALLED THE <u>MUTUAL INFO</u>.

NOTE: 1) $I(y;x) = I(x;y)$

2) IF x, y ARE INDEPENDENT, $I(x;y) = 0$

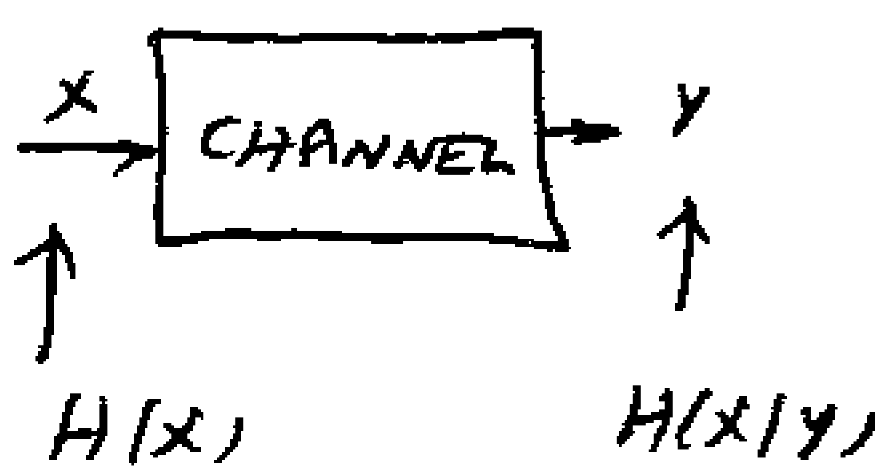
3) FROM (1)

$$I(x;y) = H(x) - H(x|y)$$
$$= H(y) - H(y|x)$$

CHANNEL CAPACITY

$$C = \max_{p(x)} I(x;y)$$

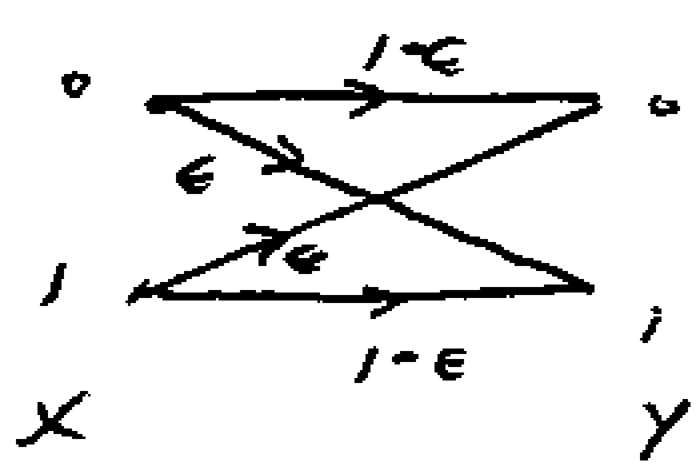
$$= \max_{p(x)} H(x) - H(x|y)$$



DIFFERENCE IS INFORMATION GAINED VIA USE OF CHANNEL.

$(H(x|y) = H(x) \Rightarrow \,?)$

EXAMPLE: BSC



$$C = \max_{p(x)} I(x;y)$$

$$= \max_{p(x)} H(y) - H(y|x) \qquad \text{(ALTERNATIVE FORM)}$$

NOW $H(Y|X) = -\sum_x \sum_y p(x,y) \log_2 p(y|x)$

$$= -\sum_x \Big[\underbrace{\sum_y p(y|x) \log_2 p(y|x)} \Big] p(x)$$

$$x = 0 \Rightarrow (1-\epsilon) \log_2 (1-\epsilon) + \epsilon \log_2 \epsilon$$

$$x = 1 \Rightarrow (1-\epsilon) \log_2 (1-\epsilon) + \epsilon \log_2 \epsilon$$

$$= -\sum_x \big((1-\epsilon) \log_2 (1-\epsilon) + \epsilon \log_2 \epsilon \big) p(x)$$

$$= -(1-\epsilon) \log_2 (1-\epsilon) - \epsilon \log_2 \epsilon = H_b(\epsilon)$$

$$C = \max_{p(x)} \; H(Y) - H_b(\epsilon)$$

$\uparrow$ $\leq 1 \Rightarrow$ FIND $p(x)$ SO THAT $p(Y=0) = p(Y=1) = \frac{1}{2}$

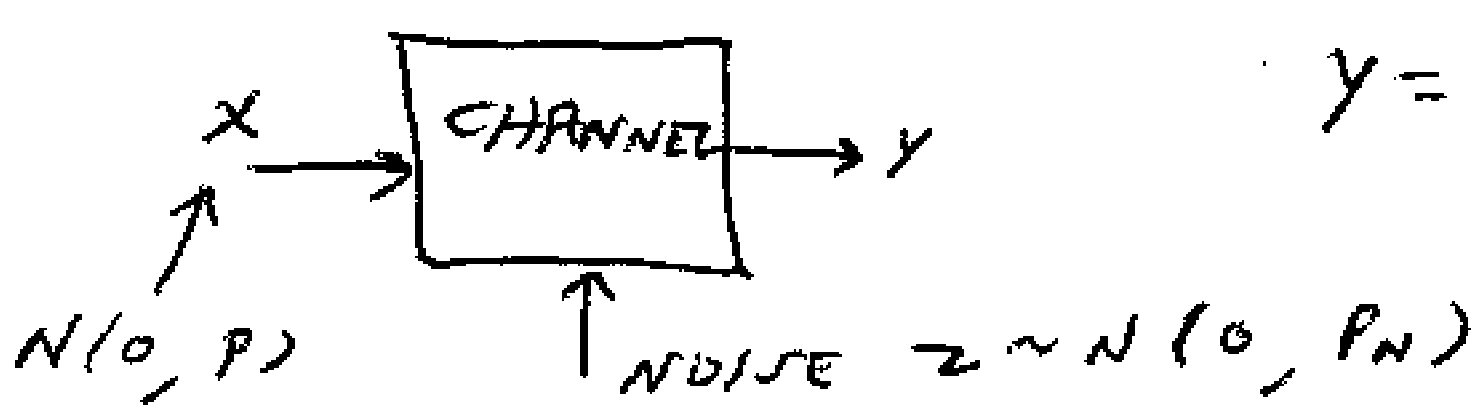
TRY $p(x) = \frac{1}{2}$ FOR $x = 0, 1$ FIND $p(y)$.

THUS, $C = 1 - H_b(\epsilon)$

NOTE: $C = 1$ IF $\epsilon = 0$ OR 1 ?

GAUSSIAN CHANNEL CAPACITY

SHANNON'S THIRD THEOREM.



$$Y = X + Z$$

SAME DEFINITION FOR CONTINUOUS RANDOM VARIABLES EXCEPT

$I(X;Y) = H(X) - H(X|Y)$

WHERE $H(X) = -\int_{-\infty}^{\infty} f(x) \ln f(x)\,dx$ NATURAL LOG (MORE CONVENIENT) $\log_2 x = \frac{\ln x}{\ln 2}$ ⇒ SCALE FACTOR ONLY

$H(X|Y) = -\iint_{-\infty}^{\infty} f(x,y) \ln f(x|y)\,dx\,dy$

NOTE: $H(X)$ DOESN'T EXIST IN GENERAL OR MAY BE NEGATIVE BUT $I(X;Y)$ WILL EXIST AND $I(X;Y) \geq 0$.

ASIDE: IF $X \sim N(0, \sigma^2)$, $H(X) = \frac{1}{2} \ln(2\pi e \sigma^2)$

PROOF: $H(X) = -E(\ln f(x))$

$$= -E\left[\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}x^2}\right)\right]$$

$$= \frac{1}{2}\ln 2\pi\sigma^2 + \frac{1}{2\sigma^2}\underbrace{E(X^2)}_{\sigma^2}$$

$$= \frac{1}{2}\ln(2\pi e \sigma^2)$$

CAN ALSO SHOW THAT FOR ALL $f(x)$ WITH $E(X^2) \leq \sigma^2$, $N(0,\sigma^2)$ MAXIMIZES $H(X)$

$$C = \max_{f(x)} I(X;Y) = \max_{f(x)} H(Y) - H(Y|X)$$

CONSIDER $Y|X = x$ FIRST

$$Y = X + Z$$

$$Y | X = x = x + Z \sim N(x, P_N) = f(y|x)$$

$$H(Y|X) = E_{X,Y}[-\ln f(y|x)]$$

$$= -E_{X,Y}\left[\ln\left(\frac{1}{\sqrt{2\pi P_N}} e^{-\frac{1}{2P_N}(y-x)^2}\right)\right]$$

$$= \frac{1}{2}\ln 2\pi P_N + \underbrace{\frac{1}{2P_N} E_{X,Y}\{\underbrace{(Y-X)^2}_{Z^2}\}}_{E_Z\{Z^2\} = P_N}$$

$$= \frac{1}{2}\ln 2\pi e P_N$$

NOT DEPENDENT ON PDF OF X OR ON $f(x)$

$$C = \max_{f(x)} H(Y) - \frac{1}{2}\ln 2\pi e P_N$$

BUT FOR A POWER CONSTRAINT ON X, SAY

$$E(X^2) \leq P \Rightarrow E\{Y^2\} = E(X^2) + E\{Z^2\} \leq P + P_N$$

$H(Y)$ IS MAXIMUM WHEN Y IS GAUSSIAN AND $Y \sim N(0, P+P_N)$. THIS IS REALIZED WHEN $X \sim N(0, P) \Rightarrow$

$$\max_{f(x)} H(Y) = H(Y) \quad \text{FOR } Y \sim N(0, P+P_N)$$

$$\Rightarrow \quad H(Y) = \frac{1}{2}\ln 2\pi e(P+P_N)$$

FINALLY FOR THE POWER LIMITED $(E(X^2) \le P)$ GAUSSIAN CHANNEL

$$C = H(Y) - H(Y/X)$$

$$= \frac{1}{2} \ln 2\pi e (P + P_N) - \frac{1}{2} \ln 2\pi e P_N$$

$$= \frac{1}{2} \ln \left(1 + \underbrace{\frac{P}{P_N}}_{SNR}\right) \quad \text{NATS / CHANNEL USE}$$

BUT $\quad \text{BITS} = \frac{\text{BITS}}{\text{NAT}} \cdot \text{NATS}$

$$= \frac{1}{\ln 2} \cdot \text{NATS}$$

$$C = \frac{1}{2} \frac{\ln (1 + P/P_N)}{\ln 2} = \frac{1}{2} \log_2 \left(1 + \frac{P}{P_N}\right)$$

BITS/CHANNEL USE

CAPACITY FORMULA

NOW ASSUME A CONTINUOUS TIME CHANNEL WITH $E(x^2(t)) \le P$, BANDWIDTH $= W$ Hz, AND AWGN WITH PSD $N_0/2$.

WE SAMPLE AT NYQUIST RATE OF $2W$ SAMPLES/SEC. ALSO, THE NOISE POWER BECOMES

$$P_N = \int_{-W}^{W} \frac{N_0}{2} \, df = N_0 W$$

$$\Rightarrow \quad C = \frac{1}{2} \log_2 \left(1 + \frac{P}{N_0 W}\right) \quad \text{BITS/SAMPLE}$$

$$\therefore \quad C = W \log_2 \left(1 + \frac{P}{N_0 W}\right) \quad \text{BITS/SEC}$$

PROBABLY THE MOST FAMOUS FORMULA IN COMM. THEORY!

EXAMPLE: TELEPHONE CHANNEL

$W = 3000$ Hz

$SNR = \frac{P}{N_0 W} = 39$ dB

$C = 3000 \log_2 (1 + 7943) = 38,867$ BITS/SEC

$\Rightarrow$ POSSIBLE TO TRANSMIT $\frac{38,867}{2400} \simeq 16$ LPC-10 VOICE CHANNELS OVER 3 KHz CHANNEL!

BOUNDS ON COMMUNICATION

FOR GAUSSIAN CHANNEL IT IS POSSIBLE TO TRANSMIT AT A RATE R BITS/SEC WITHOUT ERROR AS LONG AS $R < C = W \log_2 (1 + P/P_N)$.

NOTE THAT THE USE OF THE TERM "BITS"

REFERS TO IID RANDOM VARIABLES TAKING ON VALUES 0 OR 1 WITH EQUAL PROBABILITIES, ie, INFORMATION BITS OR A SOURCE WITH ENTROPY H(X) WHICH EMITS A SYMBOL EVERY T SEC IS EMITTING

$$R = \frac{H(x)}{T} \quad \text{BITS/SEC}$$

$\Rightarrow$ FOR ERRORLESS TRANSMISSION

$$\frac{H(x)}{T} < W \log_2(1 + P/N_0W) \quad \text{BITS/SEC}$$

EXAMPLE : FOR M-ARY ORTHOGONAL SIGNALING, FOR EXAMPLE PPM, AS $M \to \infty \Rightarrow$ EACH PPM PULSE HAS WIDTH $\to 0 \Rightarrow W \to \infty$ (SLIDE #84)

$$C = \lim_{W \to \infty} W \log_2(1 + P/N_0W)$$

$$= \frac{P}{N_0}\log_2 e = 1.44\, P/N_0$$

CAN'T INCREASE RATE INDEFINITELY IF SNR IS FIXED AS BANDWIDTH INCREASES CAN SIGNAL FASTER (NYQUIST CRITERION) BUT LET IN MORE NOISE ($P_N = N_0W$).

CONVENIENT TO DEFINE <u>SPECTRAL</u> <u>BIT</u> <u>RATE</u> AS

$$r = R/W \qquad (\text{BITS/SEC})/\text{Hz}$$

SO THAT $\quad r = \frac{R}{W} < \frac{C}{W} = \log_2\left(1 + \frac{P}{N_0 W}\right)$

ALSO, OUR STANDARD MEASURE OF SNR IS $\mathcal{E}_b/N_0$ OR SNR PER BIT.

BUT $\quad \mathcal{E}_b = (\text{AVERAGE POWER}) \times \underbrace{\text{BIT INTERVAL TIME}}_{\text{SEC/BIT}}$

$$= P \cdot \frac{1}{R}$$

$$r < \log_2\left(1 + \frac{R\,\mathcal{E}_b}{W N_0}\right) = \log_2\left(1 + r\,\mathcal{E}_b/N_0\right)$$

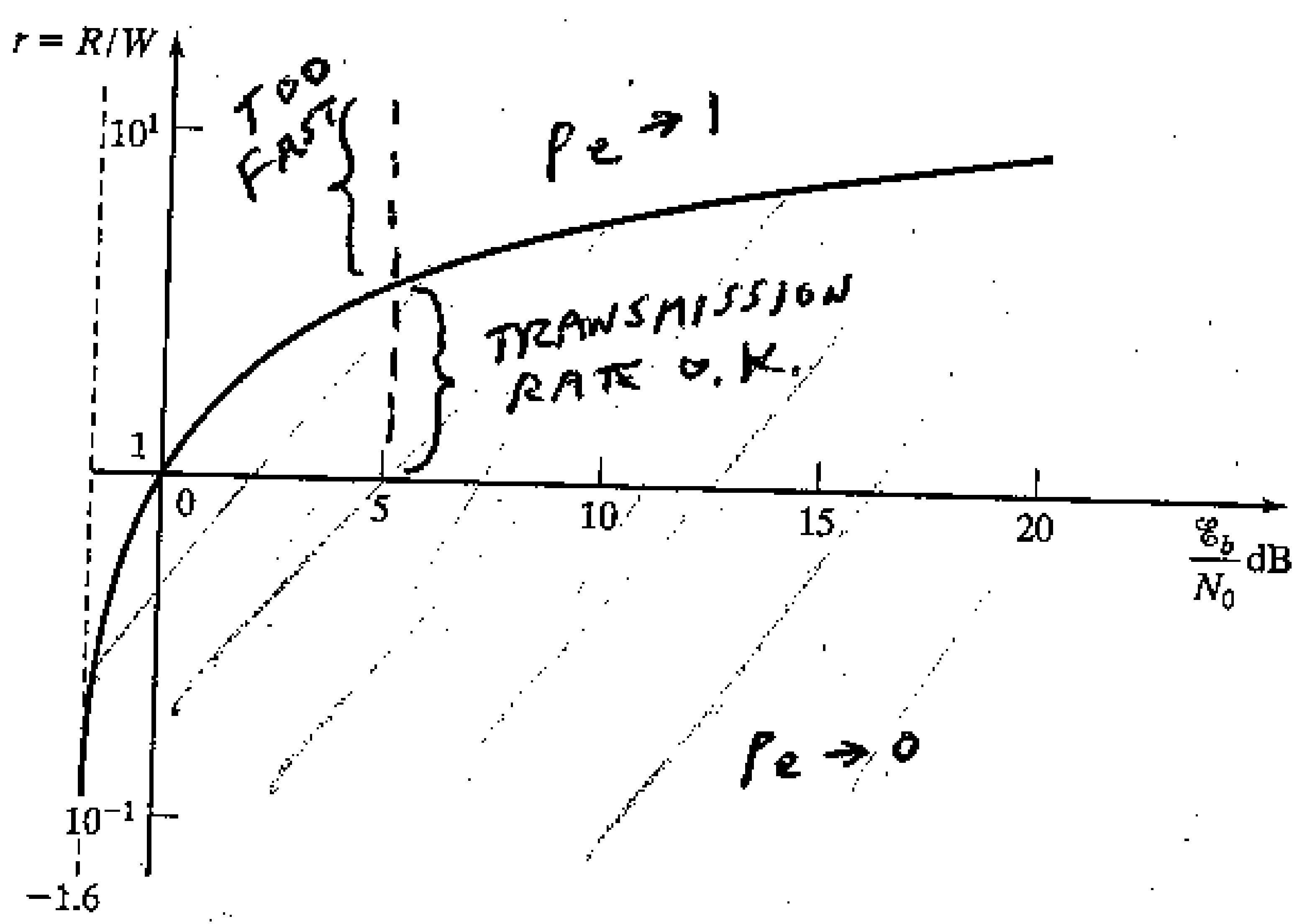


Figure 12.17 Spectral bit rate versus SNR/bit in an optimal system.

FOR $\mathcal{E}_b/N_0 < -1.6$ dB WE <u>CANNOT</u> TRANSMIT RELIABLY AT ANY RATE.

NOTE: IF W IS LARGE ⇒ r IS SMALL AND MAIN CONCERN IS POWER. CALLED POWER-LIMITED CASE (VOYAGER SPACECRAFT) USE ORTHOGONAL SIGNALING

IF W IS SMALL ⇒ r IS LARGE AND MAIN CONCERN IS BANDWIDTH. CALLED BANDWIDTH LIMITED CASE (WIRELINES). USE CROWDED CONSTELLATIONS SUCH AS PAM, QAM, ETC. ~~BUT~~ NEED GOOD SNR.

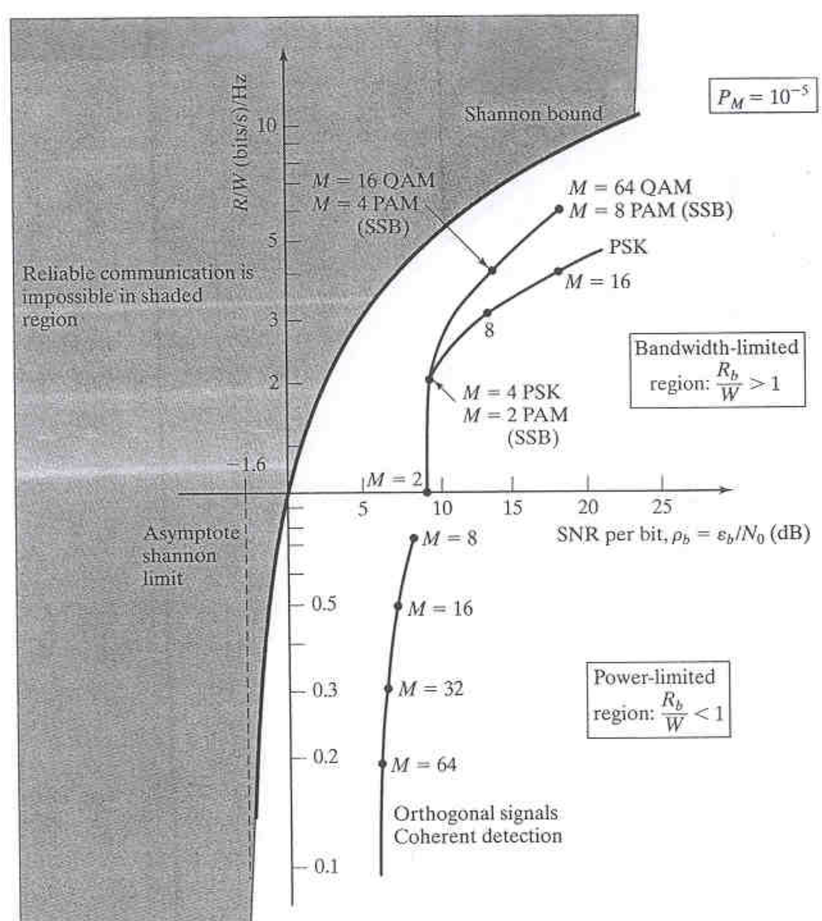


Figure 10.44 Comparison of several modulation methods at 10^{-5} symbol rate.

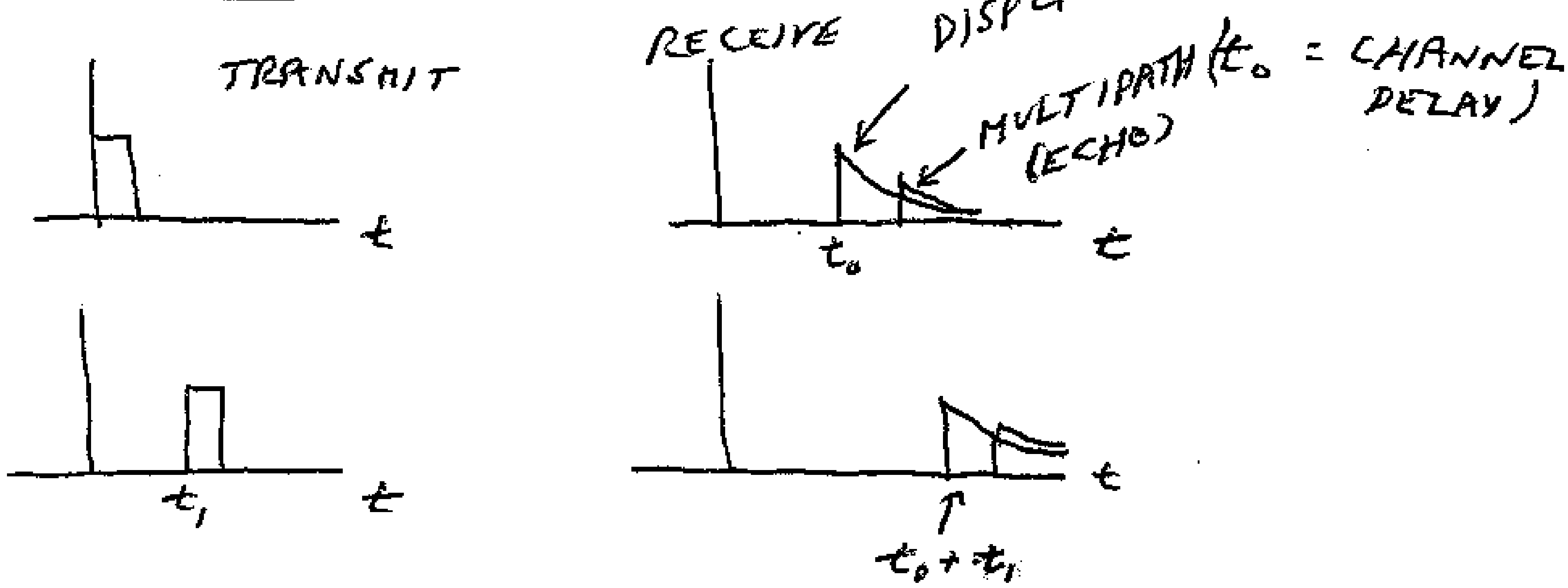
CHAPTER 11 - SELECTED TOPICS

FADING MULTIPATH CHANNELS

EXAMPLES ARE CELL PHONES (WIRELESS) AND UNDERWATER ACOUSTIC CHANNELS.

CHANNEL IS LINEAR BUT TIME VARYING (CELL PHONE IN MOVING CAR).

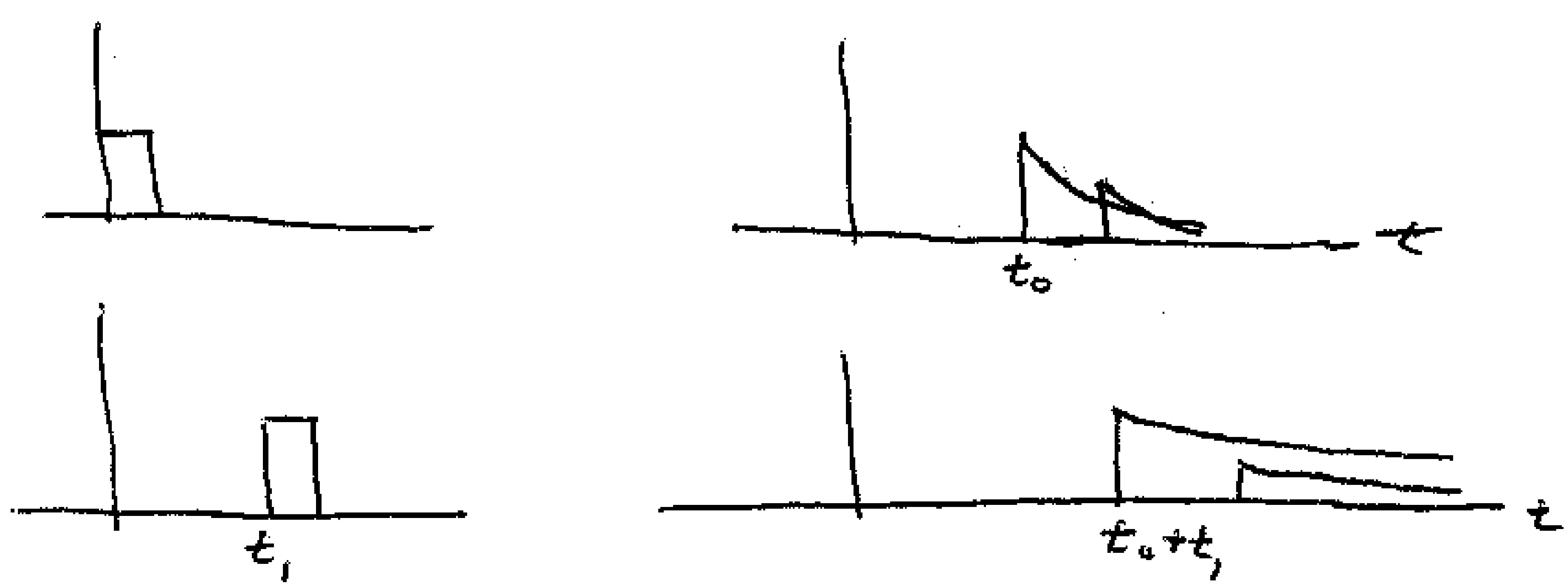
EXAMPLE: MULTIPATH



MODEL USING LTI SYSTEM, $\delta(t) \rightarrow h(t)$

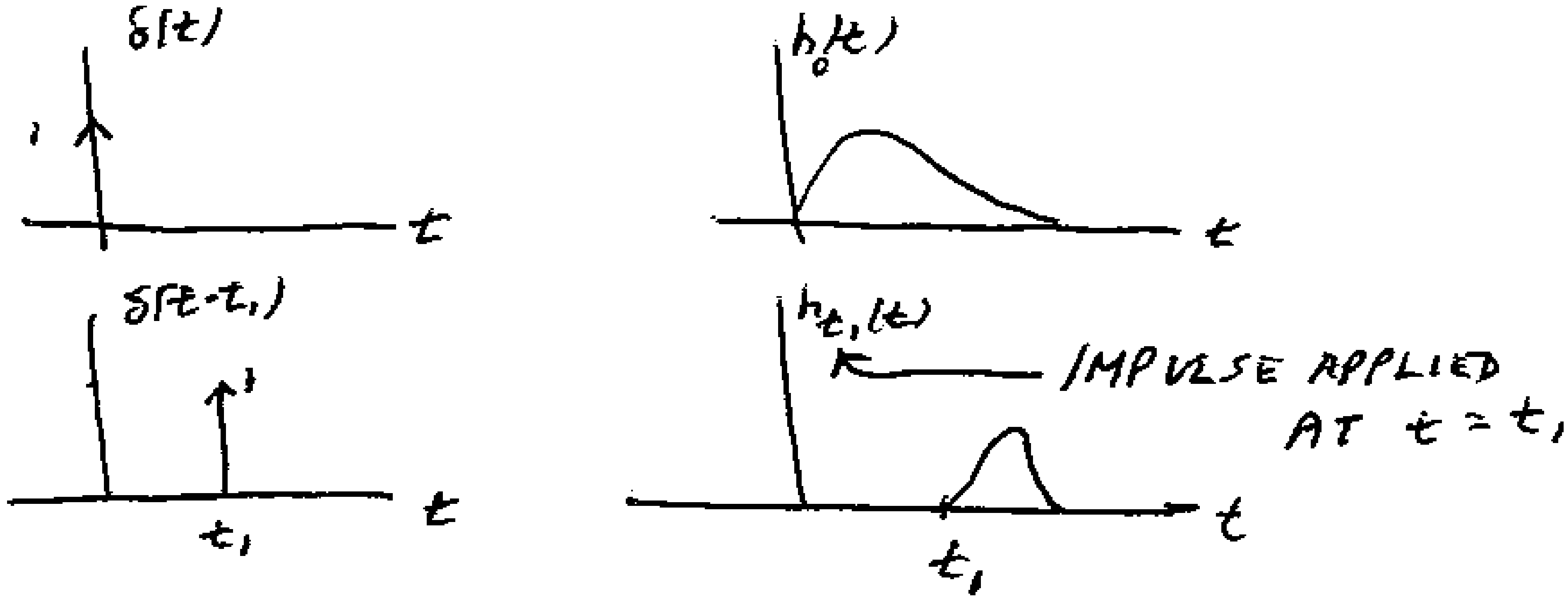
$\delta(t-t_1) \rightarrow h(t-t_1)$

EXAMPLE: FADING MULTIPATH



MODEL USING LINEAR TIME VARYING (LTV) SYSTEM, $\delta(t) \to h(t)$

$$\delta(t - t_1) \to \ ?$$



OUTPUT IS $h_{t_1}(t)$. IF LTI, $h_{t_1}(t) = h_0(t - t_1)$.

NOTE THAT $h_{t_1}(t)$ IS OUTPUT AT TIME t FOR IMPULSE APPLIED $t - t_1$ SEC EARLIER FOR LTI SYSTEM. HENCE, FOR LTI SYSTEM LET $t - t_1 = \tau$ = AGE VARIABLE = TIME SINCE IMPULSE APPLIED

OUTPUT TIME

$$h_{t_1}(t) = h_{\underbrace{t-\tau}_{\text{IMPULSE TIME (WHEN APPLIED)}}}(t)$$

OUTPUT TIME

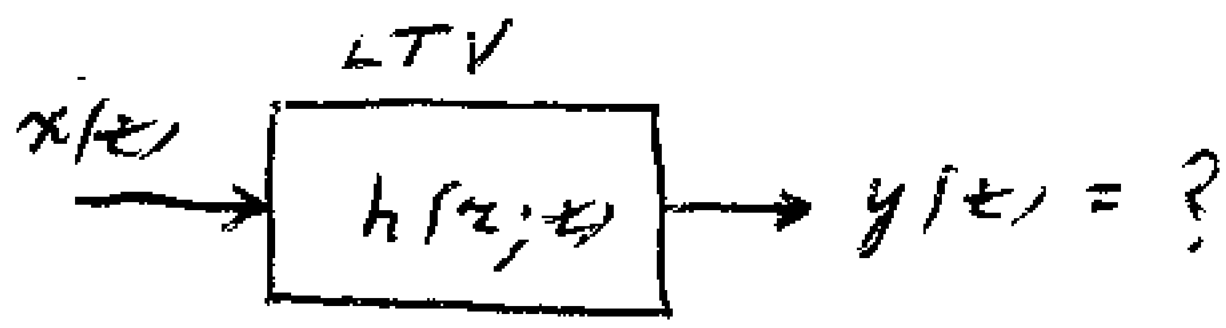
DENOTE INSTEAD BY $h(\tau; t)$

AGE TIME

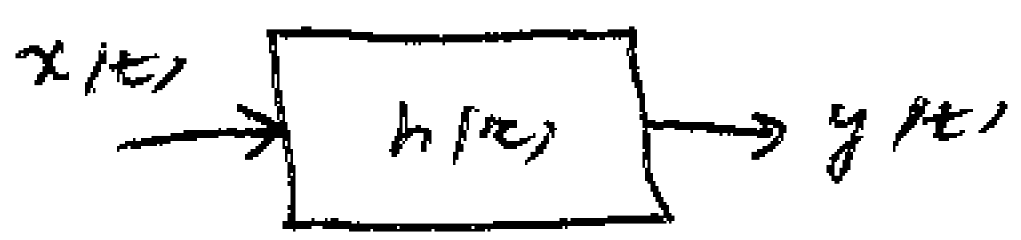
$h(\tau; t)$ = OUTPUT OF SYSTEM AT TIME t FOR IMPULSE APPLIED τ SEC EARLIER (AT $t - \tau$).

IF LTI, THEN THIS OUTPUT IS ALWAYS INDEPENDENT OF t (WHY?) BUT DEPENDENT ON τ $\Rightarrow$ $h(\tau;t) = h_{LTI}(\tau)$.

ASIDE - LTV SYSTEMS



FIRST LOOK AT LTI



$$x(t) = \int_{-\infty}^{\infty} x(\tau)\cdot\delta(t-\tau)\,d\tau$$

$\delta(t) \rightarrow h(t)$ DEFINITION

$\delta(t-\tau) \rightarrow h(t-\tau)$ TIME INVARIANT

$x(\tau)\,\delta(t-\tau) \rightarrow x(\tau)\,h(t-\tau)$ HOMOGENEITY

$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \rightarrow \underbrace{\int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau}_{y(t)}$ SUPERPOS.

LINEARITY

FOR LTV:

$\delta(t-t_1) \rightarrow h(t-t_1;t)$

$x(t_1)\,\delta(t-t_1) \rightarrow x(t_1)\,h(t-t_1;t)$ HOMOGEN.

$x(t) = \int_{-\infty}^{\infty} x(t_1)\,\delta(t-t_1)\,dt_1 \rightarrow \underbrace{\int_{-\infty}^{\infty} x(t_1)\,h(t-t_1;t)\,dt_1}_{y(t)}$ SUPER.

BUT LETTING $\tau = t - t_1 \Rightarrow t_1 = t - \tau$

(*) $$y(t) = \int_{-\infty}^{\infty} h(\tau; t)\, x(t-\tau)\, d\tau$$ ANALOGOUS RESULT TO CONVOLUTION

WE CAN ALSO DEFINE A TIME VARYING FREQ. RESPONSE AS

$$H(f;t) = \int_{-\infty}^{\infty} h(\tau;t)\, e^{-j2\pi f\tau}\, d\tau$$

(IF LTI, $h(\tau;t) = h(\tau) \Rightarrow H(f;t) = H(f)$, THE USUAL DEFINITION)

SINCE $H(f;t) = \mathcal{F}_{\tau}\{h(\tau;t)\}$

$$h(\tau;t) = \int H(f;t)\, e^{j2\pi f\tau}\, df$$

$$\Rightarrow y(t) = \int\int H(f;t)\, e^{j2\pi f\tau}\, df\, x(t-\tau)\, d\tau$$ FROM (*)

$$= \int H(f;t) \underbrace{\int x(t-\tau)\, e^{j2\pi f\tau}\, d\tau}_{\tau' = -\tau}\, df$$

$$\int x(\tau'+t)\, e^{-j2\pi f\tau'}\, d\tau' = \mathcal{F}_{\tau'}\{x(\tau'+t)\}$$

$$= X(f)\, e^{j2\pi f t}$$

$$y(t) = \int_{-\infty}^{\infty} \underbrace{H(f;t)}_{= H(f) \text{ FOR LTI}}\, X(f)\, e^{j2\pi f t}\, df$$

SUMMARY

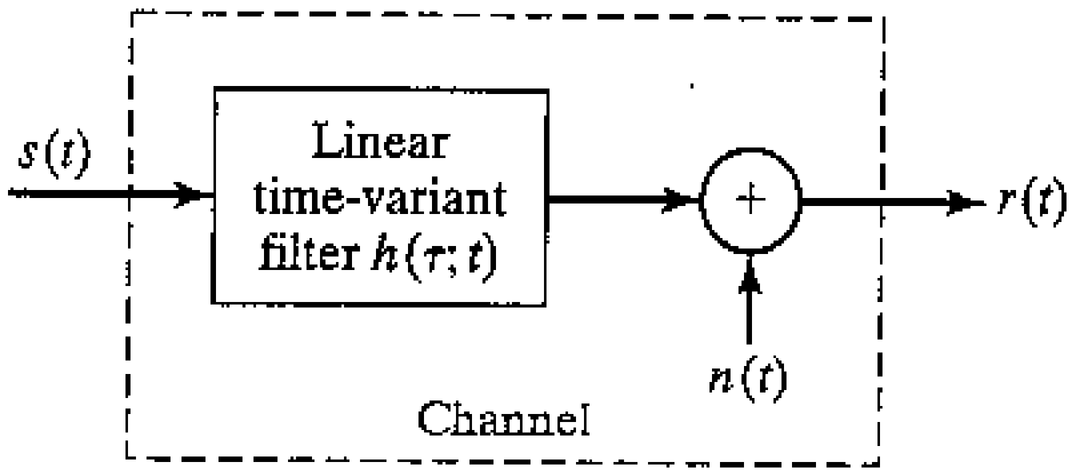


Figure 1.10 Linear time-variant filter channel with additive noise.

$$r(t) = \int_{-\infty}^{\infty} h(\tau; t)\, s(t-\tau)\, d\tau + n(t) \qquad (1)$$

$$= \int_{-\infty}^{\infty} H(f; t)\, S(f)\, e^{j2\pi f t}\, df + n(t) \qquad (2)$$

SPECIAL CASE: IF THE MULTIPATH IS DISCRETE (FINITE NUMBER OF PATHS)

$$h(\tau; t) = \sum_{n=1}^{L} \alpha_n(t)\, \delta(\tau - \tau_n(t))$$

$$\Rightarrow \quad r(t) = \sum_{n=1}^{L} \alpha_n(t)\, s(t - \tau_n(t)) + n(t) \qquad \text{FROM (1)}$$

L PATHS WITH PATH n HAVING TIME VARYING TIME DELAY $\tau_n(t)$ AND TIME VARYING GAIN $\alpha_n(t)$.

CHANNEL MODELS

THE TIME DELAYS $\tau_n(t)$ AND GAINS $\alpha_n(t)$ ARE UNKNOWN AND MUST BE DESCRIBED STATISTICALLY.

WE CONSIDER ONLY THE CASE OF A CHANNEL FOR WHICH $H(f;t)$ IS CONSTANT WITH f AT BASEBAND ($f=0$). THEN,

$$H(f;t) = H(0;t) = c(t)$$

$$r(t) = \int_{-\infty}^{\infty} H(f;t)\, S(f)\, e^{j2\pi f t}\, df + n(t)$$

$$= c(t)\, s(t) + n(t)$$

CALLED A FREQ. NONSELECTIVE CHANNEL.

(CAN ALSO BE THOUGHT OF AS $L=1$, $\alpha_1(t) = c(t)$, $\tau_1(t) = 0$)

A SIMILAR CHANNEL MODEL CAN BE USED FOR A BANDPASS SIGNAL FOR WHICH $H(f;t)$ IS CONSTANT WITH FREQ. OVER A NARROW BAND ABOUT A CARRIER FREQ. f_c. NOW $c(t)$ MUST BE COMPLEX.

EXAMPLE : TRANSMIT $s(t) = \cos 2\pi f_c t$

$$r(t) = \alpha(t)\, s(t-\tau(t)) \qquad \text{(IGNORE } n(t)\text{)}$$

$$= \alpha(t) \cos\left(2\pi f_c (t-\tau(t))\right)$$

$$= \operatorname{Re}\left[\alpha(t)\, e^{j2\pi f_c (t-\tau(t))}\right]$$

$$= \operatorname{Re}\left(\alpha(t)\, e^{-j2\pi f_c \tau(t)}\, e^{j2\pi f_c t}\right)$$

LET $c(t) = \alpha(t)\, e^{-j 2\pi f_c \tau(t)}$

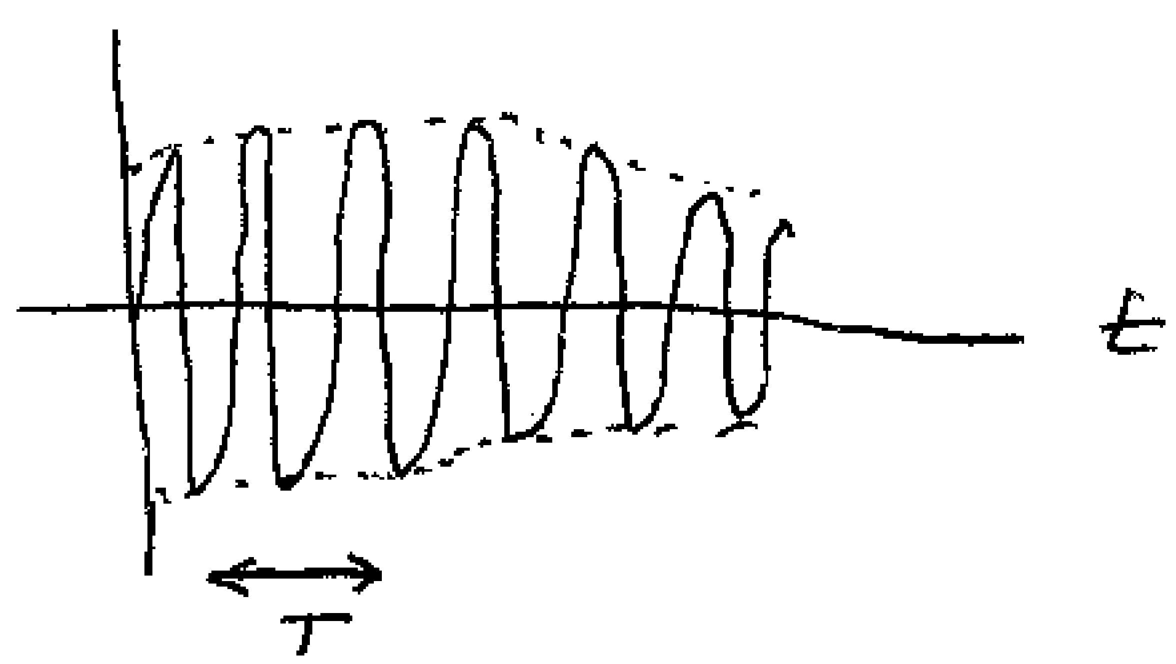
$= \alpha(t)\, e^{-j\phi(t)}$

$r(t) = Re\left(c(t)\, e^{j 2\pi f_c t} \right)$

SINCE f_c IS LARGE, $\phi(t) = 2\pi f_c \tau(t)$ IS VERY UNPREDICTABLE. USUALLY MODELED BY $\phi(t) \sim U(0, 2\pi)$. ALSO $\alpha(t)$ IS MODELED AS A RAYLEIGH RANDOM VARIABLE WITH PDF

$$f(\alpha) = \begin{cases} \dfrac{\alpha}{\sigma^2} e^{-\frac{1}{2}\alpha^2/\sigma^2} & \alpha > 0 \\ 0 & \alpha < 0 \end{cases}$$

AT ANY TIME t. FINALLY, IF THE CHANNEL IS SLOWLY FADING.



T = BIT INTERVAL

$\Rightarrow$ CAN CONSIDER $\alpha(t)$, $\phi(t)$ CONSTANT WITH t. OVER T SEC.

HENCE, SLOWLY FADING FREQ. NONSELECTIVE RAYLEIGH CHANNEL MODEL IS

$$\cos 2\pi f_c t \longrightarrow \boxed{\text{CHANNEL}} \longrightarrow \alpha \cos(2\pi f_c t + \phi) \qquad T_1 \le t \le T_1 + T$$

WHERE $\alpha \sim$ RAYLEIGH, $\phi \sim$ UNIF $\Big\}$ INDEPENDENT RANDOM VARIABLES

ANTIPODAL SIGNALS PERFORMANCE

CONSIDER BINARY PSK. WE TRANSMIT

$$u_m(t) = \sqrt{\frac{2\mathcal{E}_b}{T}} \cos(2\pi f_c t + m\pi) \qquad m = 0, 1$$

AND RECEIVE

$$r(t) = \alpha \sqrt{\frac{2\mathcal{E}_b}{T}} \cos(2\pi f_c t + m\pi + \phi) + n(t)$$
$$0 \le t \le T$$

ASSUME ϕ CAN BE ESTIMATED PERFECTLY AT RECEIVER $\Rightarrow$ USE
$\psi(t) = \sqrt{2/T} \cos(2\pi f_c t + \phi)$ TO CROSS-CORRELATE.

$$\Rightarrow \quad y = \alpha\sqrt{\mathcal{E}_b} \cos m\pi + n \qquad n \sim N(0, N_0/2)$$

AND $\alpha \sim$ RAYLEIGH. ML RECEIVER DECIDES $m = 0$ IF $y > 0$ AND $m = 1$ $y \le 0$.

FOR FIXED α WE HAVE $\mathcal{E} = (\alpha\sqrt{\mathcal{E}_b} \cos m\pi)^2$
$$= \alpha^2 \mathcal{E}_b$$