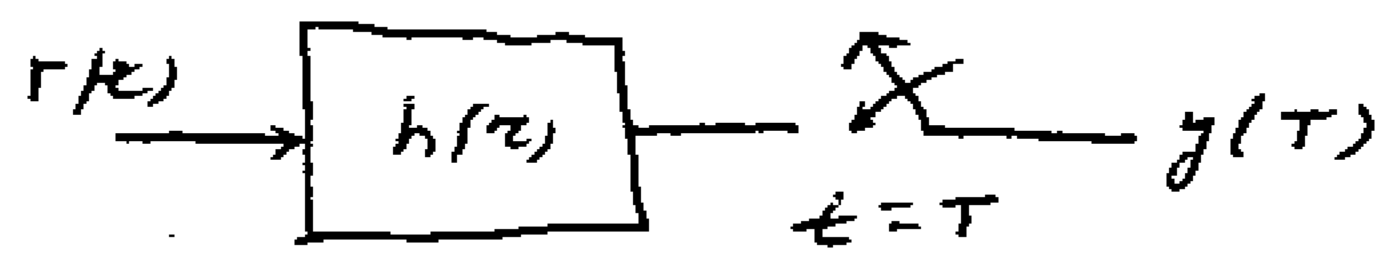
ASIDE - MATCHED FILTERS

FOR ANTIPODAL SIGNALS

$$r(t) = s_m \psi(t) + n(t) \qquad s_1 = \sqrt{E_b}$$
$$s_2 = -\sqrt{E_b}$$

IF NOISE WERE NOT PRESENT, WE COULD EASILY DECIDE WHICH SIGNAL WAS SENT. CAN WE FILTER OUT NOISE?

LET $r(t) = s(t) + n(t)$ ← AWGN $\qquad 0 \le t \le T$



CHOOSE $h(\tau)$ TO MAXIMIZE THE SIGNAL-TO-NOISE RATIO. SINCE

$$y(t) = \int_0^t r(\tau)\, h(t-\tau)\, d\tau$$
$$= \int_0^t s(\tau)\, h(t-\tau)\, d\tau + \int_0^t n(\tau)\, h(t-\tau)\, d\tau$$

$$y(T) = \underbrace{\int_0^T s(\tau)\, h(T-\tau)\, d\tau}_{y_s(T)} + \underbrace{\int_0^T n(\tau)\, h(T-\tau)\, d\tau}_{y_n(T)}$$

WANT $y_s(T) \gg y_n(T)$. DEFINE SNR AS

$$\left(\frac{S}{N}\right)_o = \frac{y_s^2(T)}{E(y_n^2(T))}$$

↑
AT OUTPUT

TO MAXIMIZE SNR:

$$\left(\frac{S}{N}\right)_0 = \frac{\left[\int_0^T s(\tau)\,h(T-\tau)\,d\tau\right]^2}{\underbrace{E\left[\left(\int_0^T n(\tau)\,h(T-\tau)\,d\tau\right)^2\right]}}$$

$$= \int\int \underbrace{E[n(\tau)n(t)]}_{\frac{N_0}{2}\delta(t-\tau)}\,h(T-\tau)\,h(T-t)\,d\tau\,dt$$

$$= \int_0^T \frac{N_0}{2} h^2(T-\tau)\,d\tau = \frac{N_0}{2}\int_0^T h^2(T-\tau)\,d\tau$$

$$\left(\frac{S}{N}\right)_0 = \frac{1}{N_0/2}\;\frac{\left[\int_0^T s(\tau)\,h(T-\tau)\,d\tau\right]^2}{\int_0^T h^2(T-\tau)\,d\tau}$$

TO MAXIMIZE USE CAUCHY-SCHWARTZ INEQUALITY:

$$\underline{x}\cdot\underline{y} = \|\underline{x}\|\,\|\underline{y}\|\cos\theta \le \|\underline{x}\|\,\|\underline{y}\| \qquad \text{EUCLIDEAN VECTORS}$$

$$\int_{-\infty}^{\infty} g_1(t)\,g_2(t)\,dt \le \sqrt{\int_{-\infty}^{\infty} g_1^2(t)\,dt}\sqrt{\int_{-\infty}^{\infty} g_2^2(t)\,dt}$$

EQUALITY HOLDS WHEN $g_1(t) = c\,g_2(t)$

LET $g_1(\tau) = s(\tau)$, $g_2(\tau) = h(T-\tau)$

$$\left[\int_0^T s(\tau)\,h(T-\tau)\,d\tau\right]^2 \le \int_0^T s^2(\tau)\,d\tau \int_0^T h^2(T-\tau)\,d\tau$$

$$\left(\frac{S}{N}\right)_0 \le \frac{1}{N_0/2}\int_0^T s^2(\tau)\,d\tau = \frac{2E_s}{N_0}$$

EQUALITY HOLDS WHEN $s(\tau) = c\,h(T-\tau) \quad 0 \le \tau \le T$

OR WHEN (LET $\tau' = T - \tau$)

$$h(\tau') = \frac{1}{c}\, s(T-\tau') \qquad 0 \le \tau' \le T$$

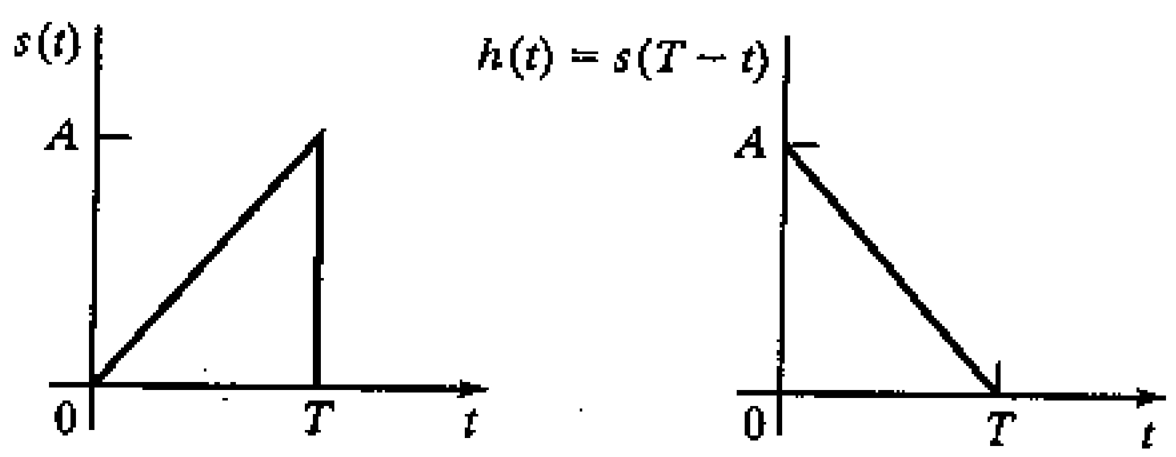
AND SINCE c IS AN ARBITRARY CONSTANT
LET $c' = 1$

$\therefore$ $h(t) = s(T-t)$ MAXIMIZES SNR AT FILTER OUTPUT

AND $\left(\frac{S}{N}\right)_{0\,MAX} = \frac{\mathcal{E}_s}{N_0/2}$

CALLED A <u>MATCHED FILTER</u>

<u>EXAMPLE</u>:



(a) Signal $s(t)$

(b) Impulse response of filter matched to $s(t)$

Figure 8.23 Signal $s(t)$ and filter matched to $s(t)$.

$= \mathcal{E}_s$

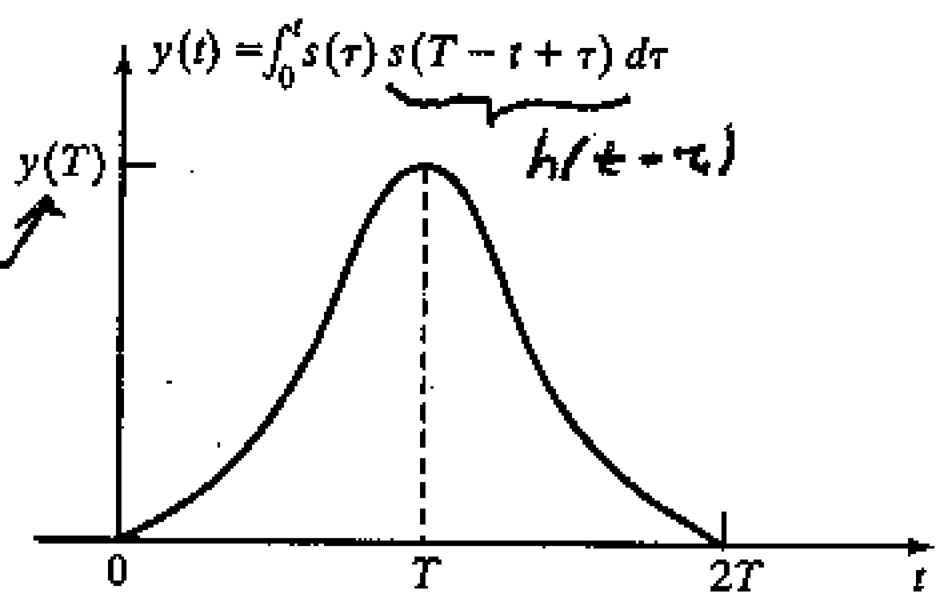
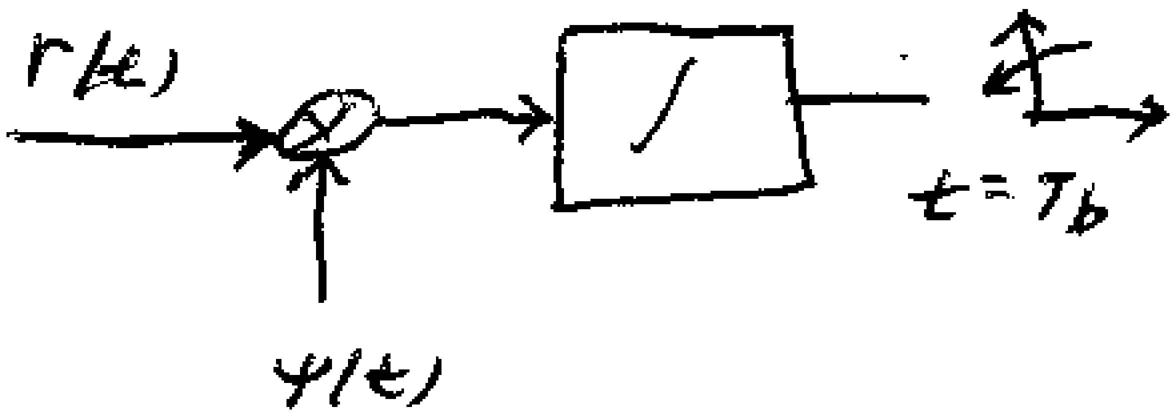


Figure 8.24 The matched filter output is the autocorrelation function of $s(t)$.

$$y(T) = \int_0^T s(\tau)\, s(\tau)\, d\tau = \mathcal{E}_s$$

IN THE TWO RECEIVERS THE DEMODULATOR IS ACTUALLY A MATCHED FILTER - GETS RID OF AS MUCH NOISE AS POSSIBLE.



$$y(T_b) = \underbrace{\int_0^t r(\tau)\psi(T_b - t + \tau)\,d\tau}_{\text{AT } t = T_b}$$

$$y(t) = \int_0^t r(\tau)\underbrace{\psi(T_b - (t-\tau))}_{h(t-\tau)}\,d\tau$$

WHERE $h(u) = \psi(T_b - u) \quad 0 \le u \le T_b$

EXAMPLE: ORTHOGONAL SIGNALS

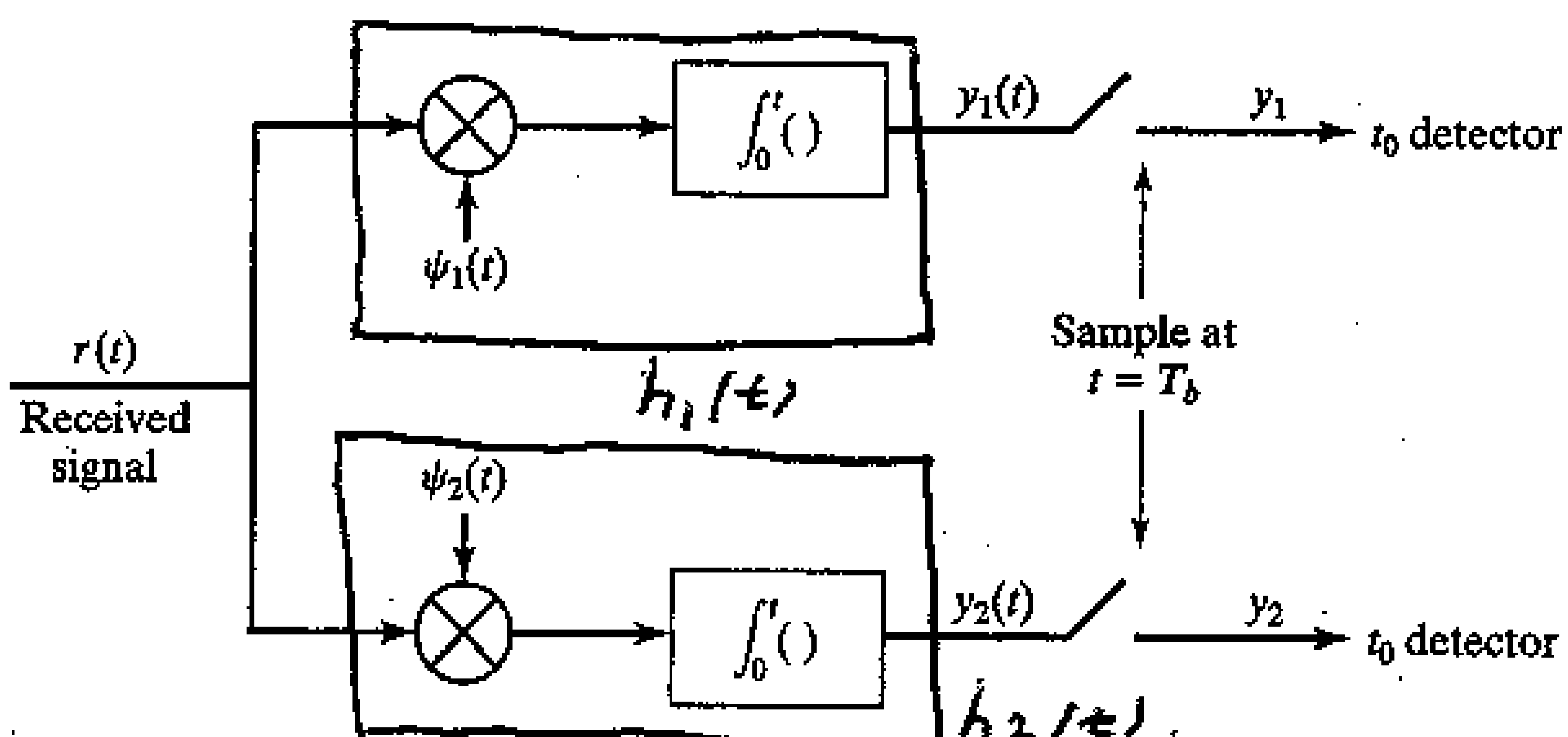
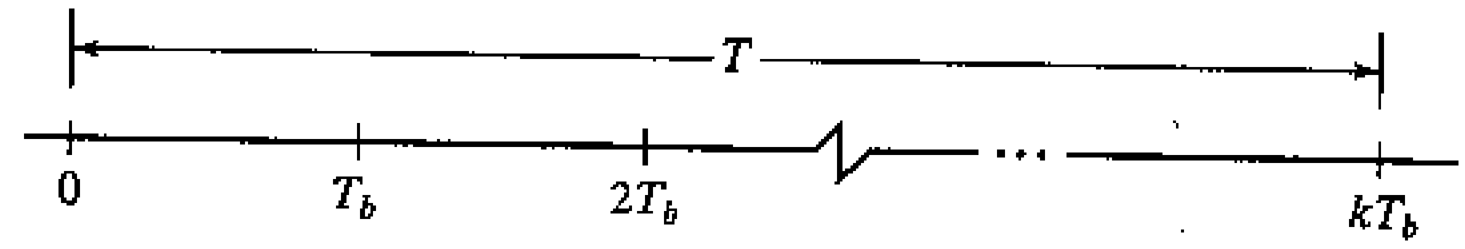


Figure 8.20 Correlation-type demodulator for binary orthogonal signals.

M-ARY PULSE MODULATION

NOW CONSIDER k-BIT BLOCKS TO BE TRANSMITTED. NEED $M = 2^k$ WAVEFORMS AND TRANSMIT ONE OF M WAVEFORMS

EVERY T SEC. TO MAINTAIN SOURCE BIT RATE $R_b = 1/T_b$, CAN HAVE $T = kT_b$



T_b = bit interval

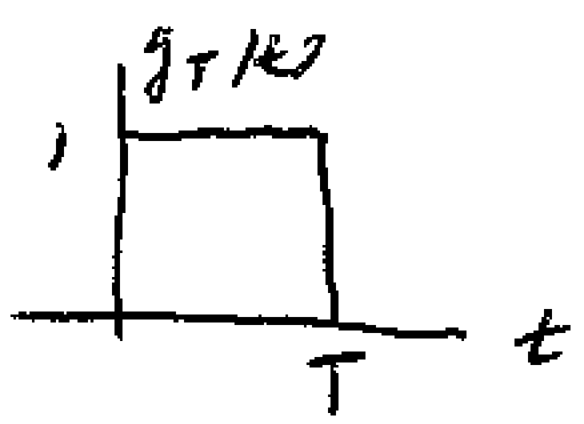
T = symbol interval = WAVEFORM (ONE OF M)

Figure 8.27 Relationship between the symbol interval and the bit interval.

PAM

NOW TRANSMIT ONE OF M LEVELS.

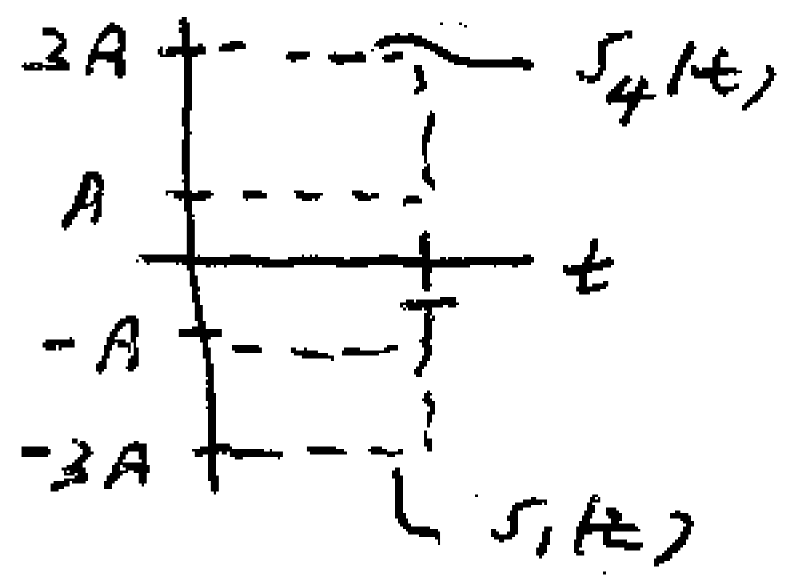
$$s_m(t) = A_m g_T(t) \qquad 0 \le t \le T$$



$$A_m = (2m-1-M)A \qquad m = 1, 2, \ldots, M$$

$$= -(M-1)A, -(M-3)A, \ldots, (M-1)A$$

$M = 4$ (2 BITS)



$M = 2 \Rightarrow$ USUAL BINARY

$$s_m(t) = \underbrace{A_m\sqrt{T}}_{s_m} \frac{g_T(t)}{\sqrt{T}} = s_m \psi(t)$$

ENERGY OF SIGNALS ARE DIFFERENT.

AVERAGE ENERGY IS (EQUIPROBABLE $P(s_m)$)

$$\mathcal{E}_{av} = \frac{1}{M}\sum_{m=1}^{M} s_m^2 = \frac{T}{M}\sum_{m=1}^{M} A_m^2 = A^2 T (M^2-1)/3$$

SINCE $s_m(t) = A_m \sqrt{T}\, \psi(t)$

$$s_m = (2m-1-M)\underbrace{A\sqrt{T}}_{d}$$

SIGNAL POINT CONSTELLATION

s_1 s_2 s_3 s_4

$-3d$ $-d$ 0 d $3d$

$M = 4$

M-ARY ORTHOGONAL SIGNALS

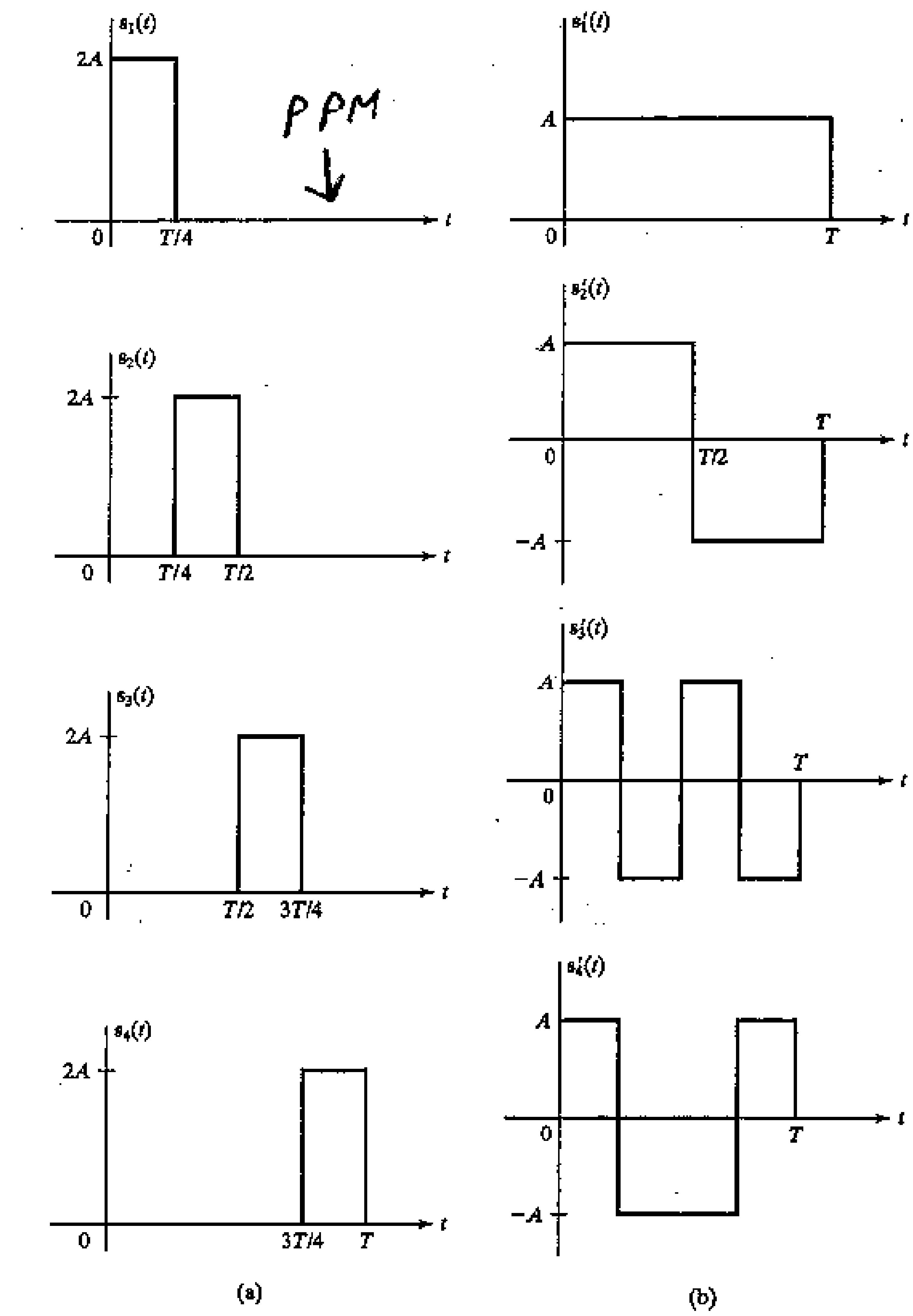


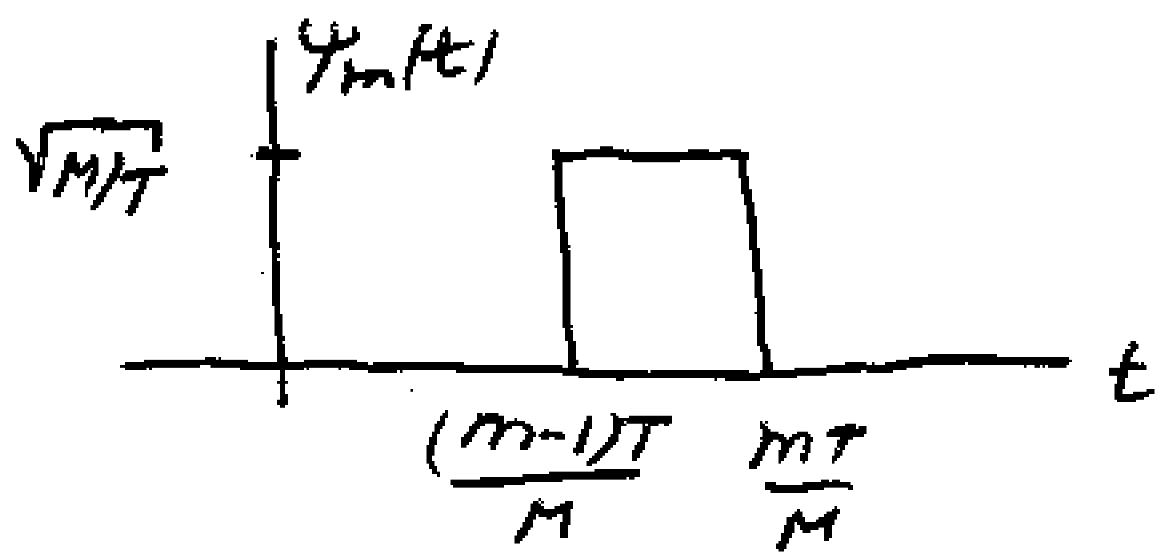
Figure 8.31 Two sets of $M = 4$ orthogonal signal waveforms.

IN EITHER CASE

$$\int_0^T s_i(t) s_j(t)\,dt = 0 \qquad i \neq j$$

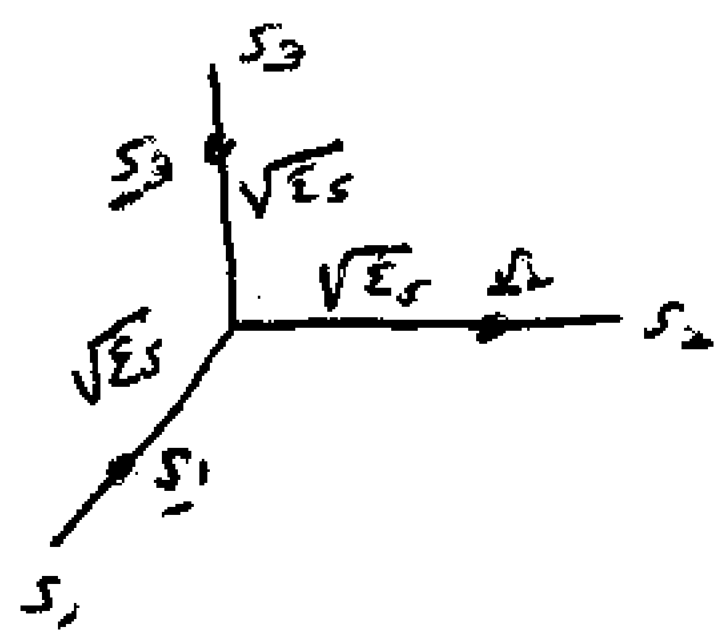
FOR M WAVEFORMS NEED $N = M$ BASIS SIGNALS $\Psi_m(t)$ (HOW ABOUT FOR M-ARY PAM?)

FOR PPM $\quad s_m(t) = \sqrt{E_s}\,\Psi_m(t) \qquad 0 \leq t \leq T$



VERIFY THIS.

$$\underline{s}_1 = (\sqrt{E_s}, 0, 0, \ldots, 0)$$
$$\underline{s}_2 = (0, \sqrt{E_s}, 0, \ldots, 0) \qquad \underline{s}_i \cdot \underline{s}_j = 0 \quad i \neq j$$
$$\vdots$$
$$\underline{s}_M = (0, 0, 0, \ldots, \sqrt{E_s})$$



$M = 3$ FOR ILLUSTRATION

ANY ROTATION ALSO YIELDS ORTHOGONAL SIGNALS

IF $\underline{s}_i \cdot \underline{s}_j = 0 \quad i \neq j$

THEN IF $\underline{U}$ IS A ROTATION MATRIX ($\underline{U}^T = \underline{U}^{-1}$)

$$\underline{s}_i \cdot \underline{s}_j = \underline{s}_i^T \underline{s}_j$$

$$\Rightarrow \quad \underline{U}\underline{s}_i \cdot \underline{U}\underline{s}_j = (\underline{U}\underline{s}_i)^T(\underline{U}\underline{s}_j) = \underline{s}_i^T \underline{U}^T \underline{U} \underline{s}_j$$

$$= \underline{s}_i^T \underline{s}_j = \underline{s}_i \cdot \underline{s}_j = 0$$

FOR $M = 2$ $\quad \underline{U} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad 0 \le \theta < 2\pi$

OPTIMUM RECEIVER - M-ARY CASE

ASSUME AWGN CHANNEL, $r(t) = s_m(t) + n(t)$

$$0 \le t \le T$$

$$s_m(t) = \sum_{k=1}^{N} s_{mk} \psi_k(t) \qquad m = 1, 2, \ldots, M$$

$$\underline{s}_m = \begin{bmatrix} s_{m1} \\ \vdots \\ s_{mN} \end{bmatrix} \quad \text{POINT IN } R^N$$

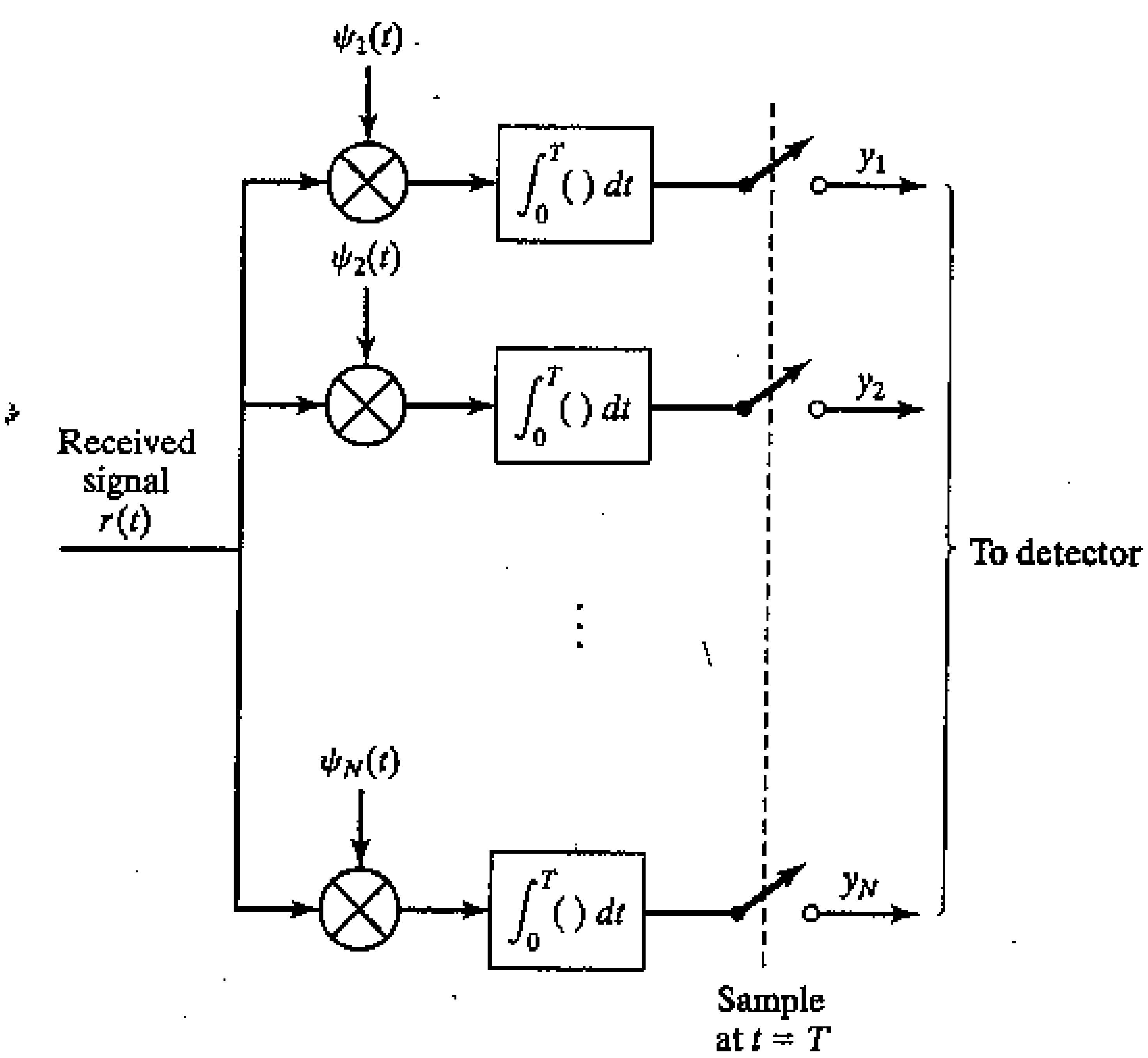


Figure 8.39 Correlation-type demodulator.

"BANK" OF CORRELATORS

IF $s_m(t)$ TRANSMITTED,

$$\underline{y} = \underline{s}_m + \underline{n}$$

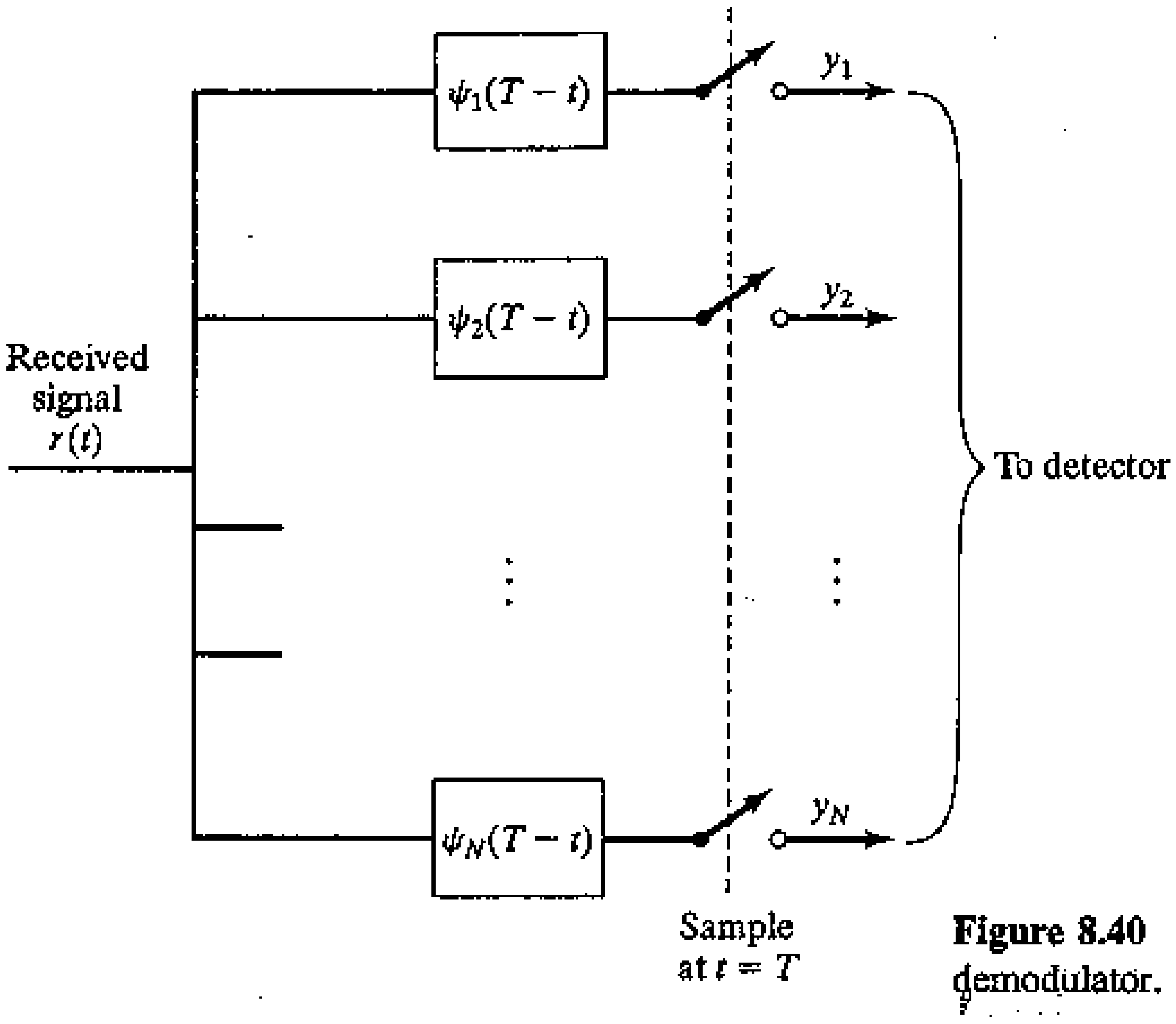


Figure 8.40 Matched-filter type demodulator.

"BANK" OF MATCHED FILTERS

TO DETERMINE OPTIMUM DETECTOR, FIRST NEED PDF OF $\underline{y}$ (TO CALCULATE P_M)

↑ PROB. OF ERROR

FOR $s_m(t)$ TRANSMITTED

$$\underline{y} = \underline{s}_m + \underline{n}$$

$$s_{mk} = \int_0^T s_m(t)\psi_k(t)\,dt$$

$$n_k = \int_0^T n(t)\psi_k(t)\,dt$$

DOESN'T DEPEND ON TRANSMITTED SIGNAL

AS BEFORE

n_k's ARE GAUSSIAN, INDEPENDENT,

$E(n_k) = 0$, $VAR(n_k) = N_0/2$

$$\Rightarrow \underline{y} \mid \underline{s}_m \sim N(\underline{s}_m, \tfrac{N_0}{2}\underline{I}_N)$$

$$f(\underline{y} \mid \underline{s}_m) = \prod_{k=1}^{N} f(y_k \mid \underline{s}_m)$$

$$= \prod_{k=1}^{N} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{1}{N_0}(y_k - s_{mk})^2}$$

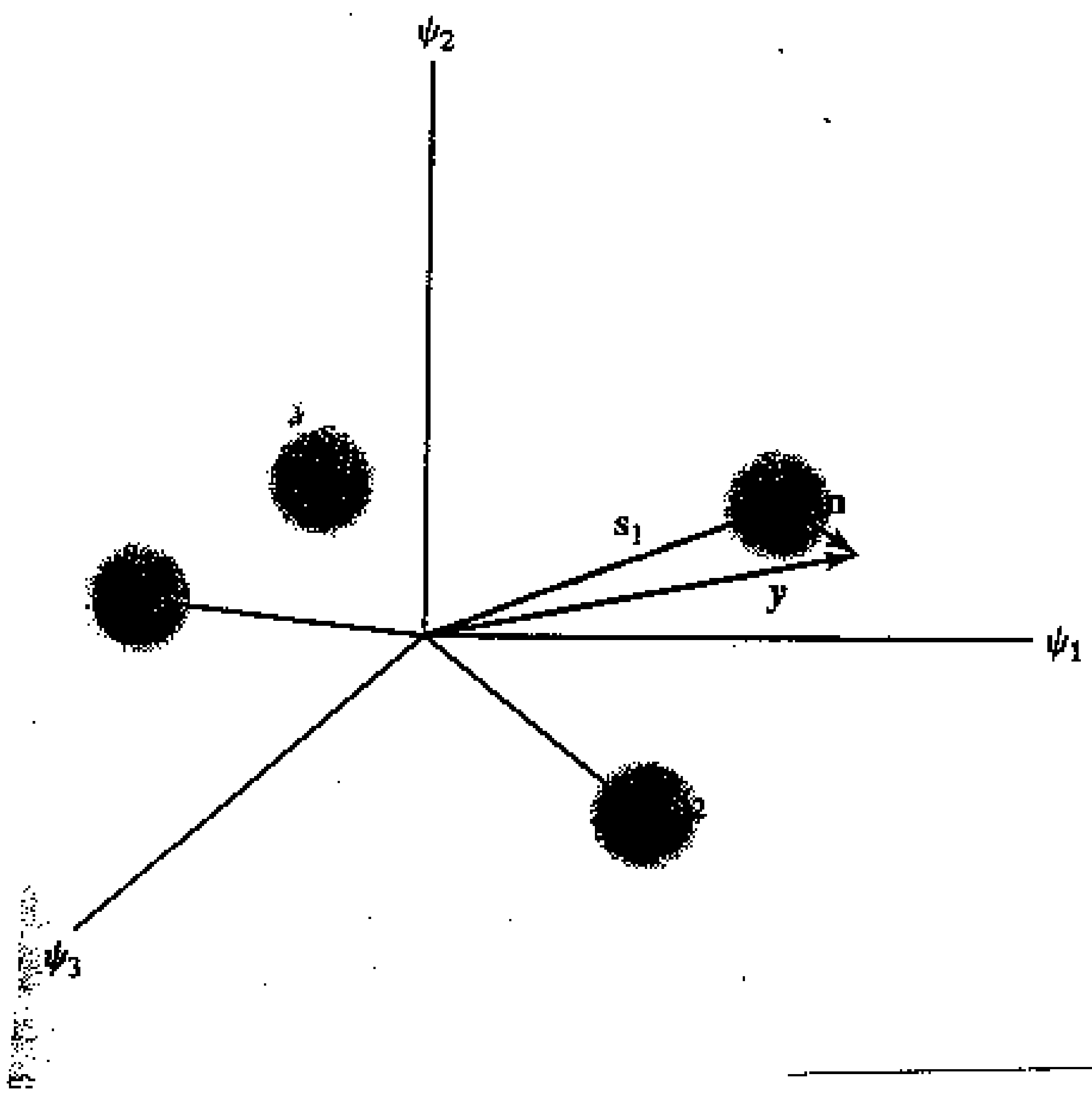
$$= \frac{1}{(\pi N_0)^{N/2}} e^{-\frac{1}{N_0}\sum_{k=1}^{N}(y_k - s_{mk})^2}$$

$$= \frac{1}{(2\pi N_0/2)^{N/2}} e^{-\frac{1}{2(N_0/2)} \underbrace{(\underline{y}-\underline{s}_m)^T(\underline{y}-\underline{s}_m)}_{\|\underline{y}-\underline{s}_m\|^2}}$$

<u>EXAMPLE</u> : $N=3$, $M=4$

$\Rightarrow$ $\underline{y}$ IS 3×1

4 SIGNALS $\Rightarrow$ $\underline{s}_1, \underline{s}_2, \underline{s}_3, \underline{s}_4$



$\underline{n} \sim N(\underline{0}, \frac{N_0}{2} \underline{I}_3)$

NOTE DISTRIBUTION OF "CLOUDS"

Figure 8.43 Signal constellation, noise cloud, and received vector for $N = 3$ and $M = 4$. It is assumed that s_1 is transmitted.

TO FIND OPTIMUM DETECTOR, LET

$P(\underline{s}_m)$ BE A PRIORI PROB. OF $s_m(t)$ TRANSMITTED

$f(\underline{y} \mid \underline{s}_m)$ BE CONDITIONAL PDF OF $\underline{y}$ GIVEN

$s_m(t)$ TRANSMITTED. BY BAYES RULE

$$P(\underline{s}_m \mid \underline{y}) = \frac{f(\underline{y} \mid \underline{s}_m)\, P(\underline{s}_m)}{\sum_{m=1}^{M} f(\underline{y} \mid \underline{s}_m)\, P(\underline{s}_m)}$$

$$= \frac{f(\underline{y} \mid \underline{s}_m)\, P(\underline{s}_m)}{f(\underline{y})}$$

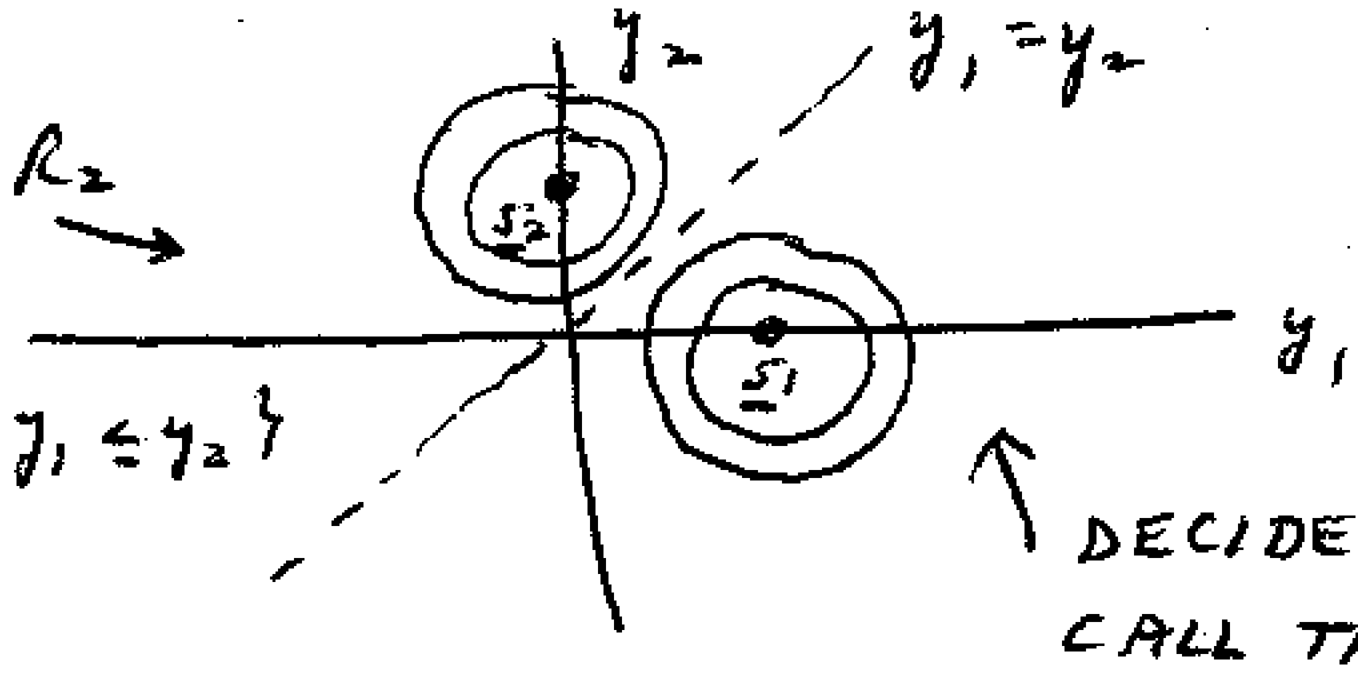
$f(\underline{y})$ ← DOESN'T DEPEND ON $\underline{s}_m$.

CALLED THE A POSTERIORI PROBABILITY - AFTER $\underline{y}$ IS RECEIVED - WILL BE DIFFERENT THAN $P(\underline{s}_m)$.

OPTIMAL DETECTOR CHOOSES SIGNAL FOR WHICH $P(\underline{s}_m \mid \underline{y})$ IS MAXIMUM. SINCE AFTER DATA IS RECEIVED IT IS MOST PROBABLE.

CALLED A MAXIMUM A POSTERIORI (MAP) DETECTOR (OR RECEIVER).

PROOF: RECALL SLIDE 59 $M = N = 2$



$R_2 = \{\underline{y} : y_1 \le y_2\}$

DECIDE $\underline{s}_1$ IF $y_1 > y_2$ AND $\underline{s}_2$ OTHERWISE

↑ DECIDE $\underline{s}_1$ HERE

CALL THIS $R_1 = \{\underline{y} : y_1 > y_2\}$

$R_1 \cup R_2$ = ENTIRE PLANE

A DECISION RULE MAPS EVERY VALUE OF $\underline{y}$ INTO R_1 OR R_2 BUT NOT BOTH.

PROB OF ERROR $= P_2$

$$= P(e|\underline{S}_1)P(\underline{S}_1) + P(e|\underline{S}_2)P(\underline{S}_2)$$

WHEN $\underline{y} \in R_1$ DECIDE $\underline{S}_1$ $\Rightarrow$ ERROR IF $\underline{S}_1$ AND $\underline{y} \in R_2$
$\underline{y} \in R_2$ DECIDE $\underline{S}_2$ $\Rightarrow$ " " $\underline{S}_2$ " $\underline{y} \in R_1$

$$P_2 = P(\underline{y} \in R_2|\underline{S}_1)P(\underline{S}_1) + P(\underline{y} \in R_1|\underline{S}_2)P(\underline{S}_2)$$

$$= \int_{R_2} f(\underline{y}|\underline{S}_1)\,d\underline{y}\; P(\underline{S}_1) + \int_{R_1} f(\underline{y}|\underline{S}_2)\,d\underline{y}\; P(\underline{S}_2)$$

$$= \int_{R_2} f(\underline{y}|\underline{S}_1)P(\underline{S}_1)\,d\underline{y} + \int_{R_1} f(\underline{y}|\underline{S}_2)P(\underline{S}_2)\,d\underline{y}$$

WE GET TO DECIDE FOR EACH $\underline{y}$ WHETHER TO PUT IT IN R_1 OR IN R_2 BUT NOT BOTH. IF WE PUT $\underline{y} = \underline{y}_0$ IN R_2, THEN P_2 INCREASES BY $f(\underline{y}_0|\underline{S}_1)P(\underline{S}_1)$. IF WE PUT IT IN R_1 P_2 INCREASES BY $f(\underline{y}_0|\underline{S}_2)P(\underline{S}_2)$. THUS, ASSIGN $\underline{y}_0$ TO R_2 IF

$$f(\underline{y}_0|\underline{S}_1)P(\underline{S}_1) < \underbrace{f(\underline{y}_0|\underline{S}_2)P(\underline{S}_2)}_{\text{LARGER}}$$

ASSIGN $\underline{y}_0$ TO R_1 IF

$$f(\underline{y}_0|\underline{S}_2)P(\underline{S}_2) < \underbrace{f(\underline{y}_0|\underline{S}_1)P(\underline{S}_1)}_{\text{LARGER}}$$

OR ASSIGN $\underline{y}$ TO R_i FOR WHICH

$$f(\underline{y} \mid \underline{s}_m) P(\underline{s}_m) \text{ IS MAXIMUM } \quad m = 1, 2$$

$$\frac{f(\underline{y} \mid \underline{s}_m) P(\underline{s}_m)}{f(\underline{y})} = P(\underline{s}_m \mid \underline{y})$$

↑ MAP RULE

NOTE: IF $P(\underline{s}_m) = 1/M$ ALL m,

$$f(\underline{y} \mid \underline{s}_m) P(\underline{s}_m) = f(\underline{y} \mid \underline{s}_m) \frac{1}{M}$$

$\Rightarrow$ DECIDE $\underline{s}_m$ FOR WHICH $f(\underline{y} \mid \underline{s}_m)$ IS MAXIMUM. CALLED THE <u>MAXIMUM LIKELIHOOD</u> (ML) RULE

NOW ASSUME EQUAL PRIOR PROBS. $\Rightarrow$ ML RULE

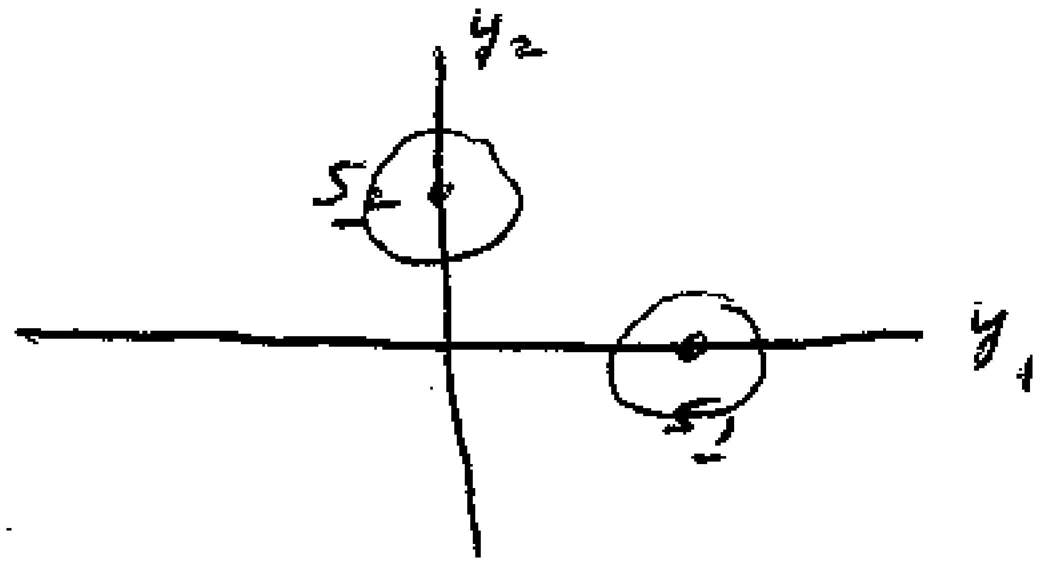
$$f(\underline{y} \mid \underline{s}_m) = \frac{1}{(2\pi \frac{N_0}{2})^{N/2}} e^{-\frac{1}{2(N_0/2)} \|\underline{y} - \underline{s}_m\|^2}$$

THIS IS MAXIMIZED OVER $\underline{s}_m$ BY CHOOSING THE SIGNAL <u>MINIMIZING</u>

$$D(\underline{y}, \underline{s}_m) = \|\underline{y} - \underline{s}_m\|^2 = \sum_{k=1}^{N} (y_k - s_{mk})^2$$

WHICH IS THE SQUARED EUCLIDEAN DISTANCE FROM $\underline{y}$ TO $\underline{s}_m$.

<u>EXAMPLE</u> : BINARY ORTHOGONAL SIGNALS (SLIDE 59)



$$\underline{s}_1 = \begin{pmatrix} \sqrt{\mathcal{E}_b} \\ 0 \end{pmatrix} \qquad \underline{s}_2 = \begin{pmatrix} 0 \\ \sqrt{\mathcal{E}_b} \end{pmatrix}$$

$P(\underline{s}_1) = P(\underline{s}_2)$

DECIDE $\underline{s}_1$ IF $\|\underline{y} - \underline{s}_1\|^2 < \|\underline{y} - \underline{s}_2\|^2$

$$(\underline{y} - \underline{s}_1)^T(\underline{y} - \underline{s}_1) < (\underline{y} - \underline{s}_2)^T(\underline{y} - \underline{s}_2)$$

$$\underline{y}^T\underline{y} - \underline{y}^T\underline{s}_1 - \underline{s}_1^T\underline{y} + \underline{s}_1^T\underline{s}_1 < \underline{y}^T\underline{y} - \underline{y}^T\underline{s}_2 - \underline{s}_2^T\underline{y} + \underline{s}_2^T\underline{s}_2$$

$$-2\underline{y}^T\underline{s}_1 + \underbrace{\underline{s}_1^T\underline{s}_1}_{\mathcal{E}_b} < -2\underline{y}^T\underline{s}_2 + \underbrace{\underline{s}_2^T\underline{s}_2}_{\mathcal{E}_b}$$

$$-2\underline{y}^T(\underline{s}_1 - \underline{s}_2) < 0$$

$$\underline{y}^T(\underline{s}_1 - \underline{s}_2) > 0$$

$$[y_1 \; y_2] \begin{bmatrix} \sqrt{\mathcal{E}_b} \\ -\sqrt{\mathcal{E}_b} \end{bmatrix} > 0$$

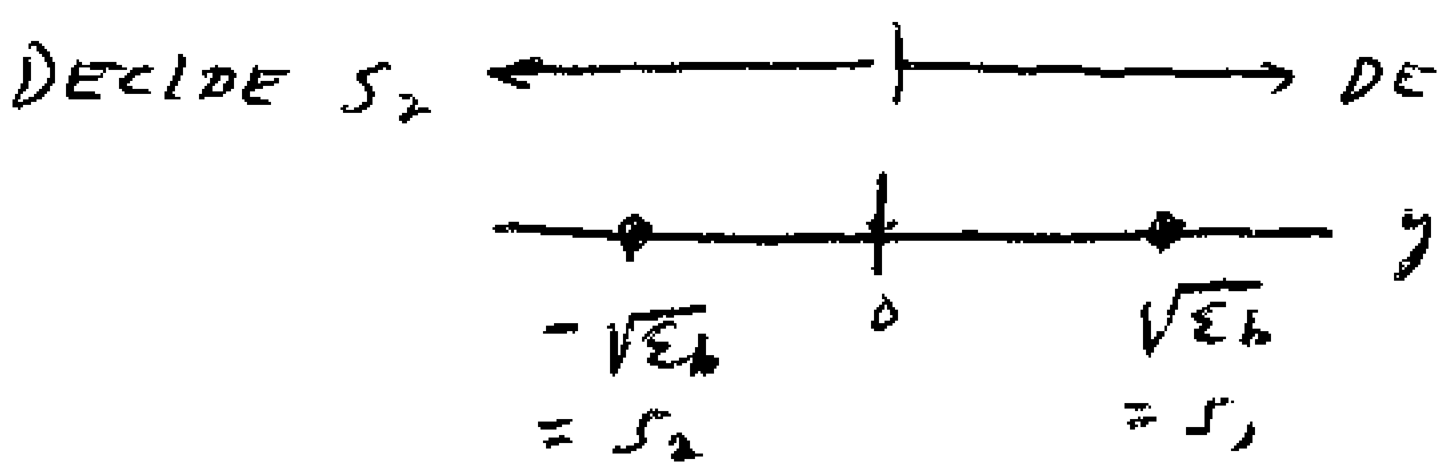
$$y_1\sqrt{\mathcal{E}_b} - y_2\sqrt{\mathcal{E}_b} > 0$$

$\therefore \quad y_1 > y_2$ IF $y_1 = y_2$, CAN DECIDE EITHER SIGNAL.

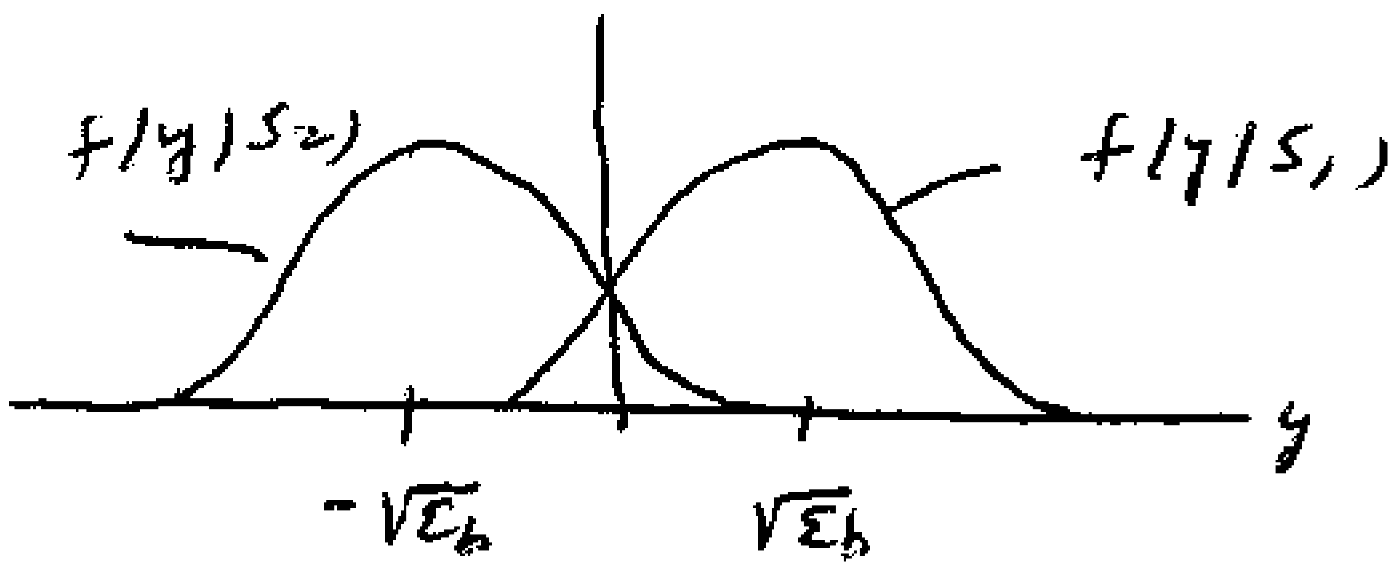
EXAMPLE : BINARY PAM $P(s_1) = P(s_2)$

$$f(y \mid s_m) = \frac{1}{\sqrt{2\pi \frac{N_0}{2}}} e^{-\frac{1}{2 N_0/2}(y - s_m)^2}$$

ML RULE $\Rightarrow$ MINIMIZE $(y - s_m)^2$



OR

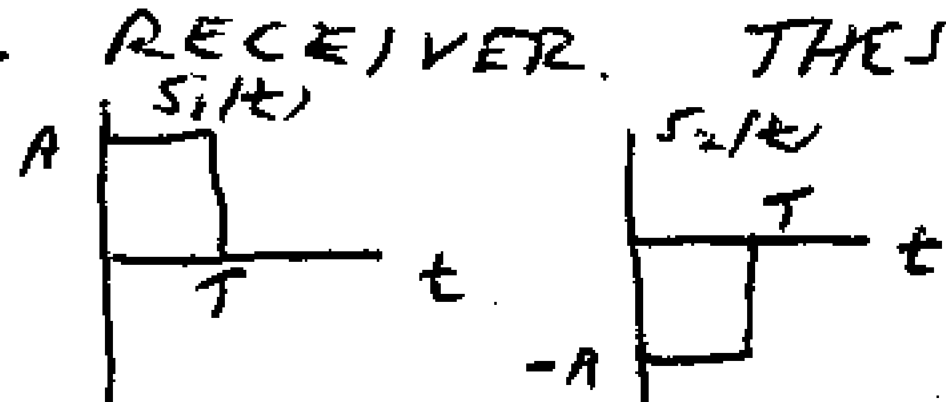


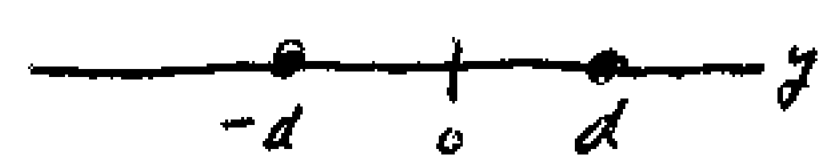
DECIDE s_1 IF $y > 0$ AND s_2 IF $y \leq 0$

WHY ?

PROB. ERROR - M-ARY PAM

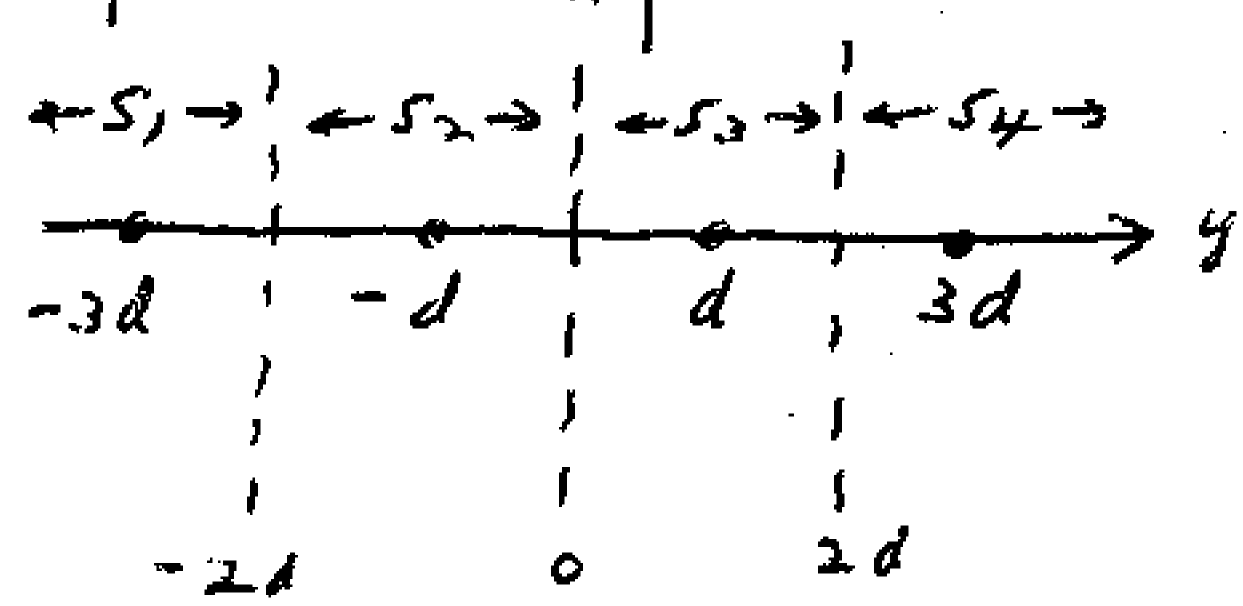
FOR $M = 2$ (BINARY), DERIVED $P_2 = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$ FOR OPTIMAL RECEIVER. THESE SIGNALS WERE ANTIPODAL





$d = A\sqrt{T}$
$= \sqrt{E_b}$

FOR $M = 4$



ML RULE

NOTE THAT P_4 IS THE PROBABILITY OF INCORRECTLY DECIDING UPON 1 OF 4 SIGNALS $\Rightarrow$ IT IS A SYMBOL ERROR, NOT A BIT ERROR AS FOR $M=2$.

SYMBOL ERROR $\Rightarrow$ ONE BIT INCORRECT (EITHER ONE) OR TWO BITS INCORRECT

$$P_4 = \frac{1}{4}\Big[\int_{-2d}^{\infty} f(y|s_1)\,dy + \int_{-\infty}^{-2d} f(y|s_2)\,dy + \int_{0}^{\infty} f(y|s_2)\,dy$$
$$+ \int_{-\infty}^{0} f(y|s_3)\,dy + \int_{2d}^{\infty} f(y|s_2)\,dy + \int_{-\infty}^{2d} f(y|s_4)\,dy\Big]$$

$$= \frac{1}{4}\Big[\, Q\left(\frac{-2d+3d}{\sqrt{N_0/2}}\right) + 1 - Q\left(\frac{-2d+d}{\sqrt{N_0/2}}\right) + Q\left(\frac{0+d}{\sqrt{N_0/2}}\right)$$
$$+ 1 - Q\left(\frac{0-d}{\sqrt{N_0/2}}\right) + Q\left(\frac{2d-d}{\sqrt{N_0/2}}\right) + 1 - Q\left(\frac{2d-3d}{\sqrt{N_0/2}}\right)\Big]$$

$$= \frac{1}{4}\Big[Q(\alpha) + 1 - Q(-\alpha) + Q(\alpha) + 1 - Q(-\alpha) + Q(\alpha) + 1 - Q(-\alpha) \Big] \qquad \alpha = \frac{d}{\sqrt{N_0/2}}$$

$$= \frac{1}{4}\Big[3\,Q(\alpha) + \underbrace{3 - 3\,Q(-\alpha)}_{3Q(\alpha)} \Big] = \frac{3}{2}\,Q(\alpha)$$

$$= \frac{3}{2}\,Q\left(\sqrt{\frac{2d^2}{N_0}}\right)$$

NOTE THAT T IS FIXED, IF WE CONSTRAIN AVERAGE OF THE TRANSMITTED SIGNAL, THEN $P_{AV} = E_{AV}\,T \Rightarrow E_{AV}$ MUST BE FIXED.

$$\mathcal{E}_{AV} = \frac{1}{4}(9d^2 + d^2 + d^2 + 9d^2)$$
$$= 5d^2$$

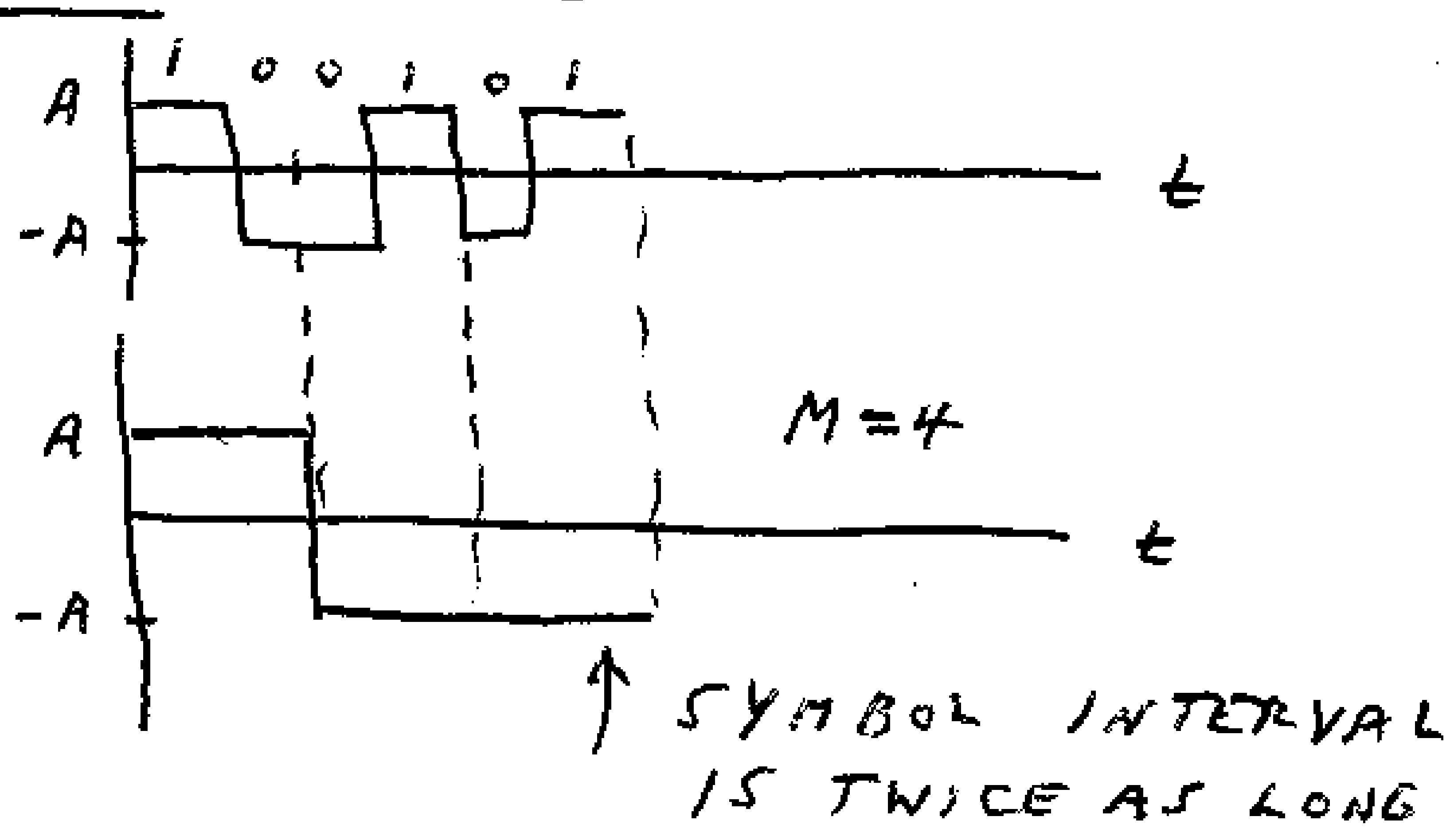
$$P_4 = \frac{3}{2}\, \varphi\left(\sqrt{\frac{2\mathcal{E}_{AV}}{5N_0}}\right)$$

FINALLY, WE EXPRESS THIS NOT WITH $\mathcal{E}_{AV}$ BUT SNR/BIT TO ALLOW BETTER COMPARISON TO BINARY CASE.

$$\mathcal{E}_{bAV} = \frac{\mathcal{E}_{AV}}{2}$$

$$P_4 = \frac{3}{2}\, \varphi\left(\sqrt{\frac{4\mathcal{E}_{bAV}}{5N_0}}\right)$$

EXAMPLE: $M = 2$



00 → −3A
01 → −A
10 → A
11 → 3A

WANT AVERAGE ENERGY OVER SAME TIME INTERVAL.

IN GENERAL $$P_M = \frac{2(M-1)}{M}\, \varphi\left(\sqrt{\frac{(6\log_2 M)\,\mathcal{E}_{bAV}}{(M^2-1)N_0}}\right)$$

(SEE BOOK)

UP TO M BITS IN ERROR

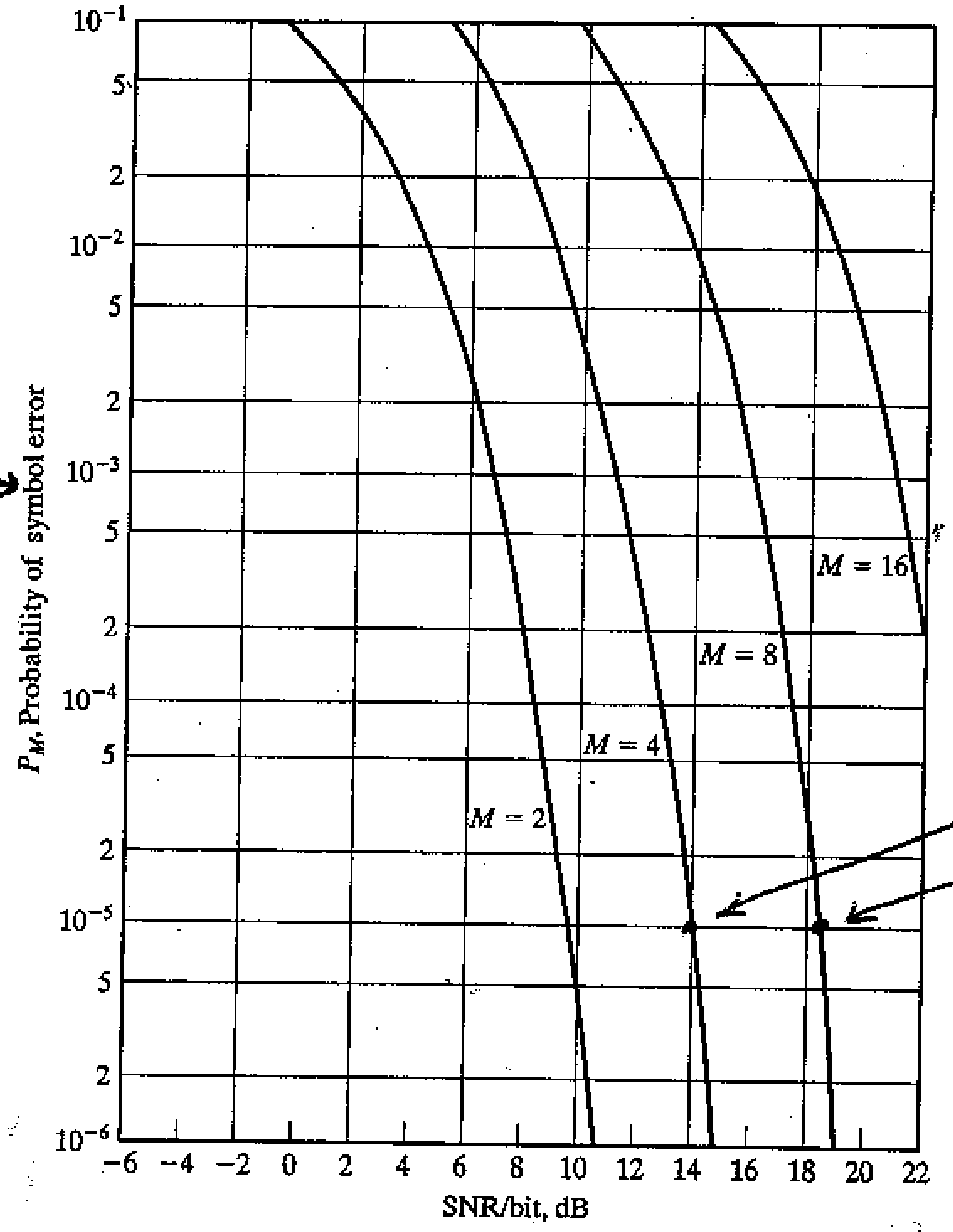


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Figure 8.45 Probability of a symbol error for PAM.

NEED ABOUT 6 dB MORE SNR/BIT FOR EACH DOUBLING OF PAM LEVELS

PROB. ERROR - M-ARY ORTHOGONAL SIGNALS

RECALL $M=2$ CASE (SLIDE 55)

$$f(y_1 | \underline{s}_1) = \frac{1}{\sqrt{2\pi N_0/2}} e^{-\frac{1}{2 N_0/2}(y_1 - \sqrt{E_b})^2}$$

$$f(y_2 | \underline{s}_1) = \frac{1}{\sqrt{2\pi N_0/2}} e^{-\frac{1}{2 N_0/2} y_1^2}$$

(ASSUMES s_1 WAS TRANSMITTED.)

IN GENERAL, IF S_1 IS TRANSMITTED,

$$\left.\begin{array}{l} y_1 \sim N(\sqrt{\mathcal{E}_s}, N_0/2) \\ y_2 \sim N(0, N_0/2) \\ \vdots \\ y_M \sim N(0, N_0/2) \end{array}\right\}$$ ALL RANDOM VARIABLES ARE INDEPENDENT

FOR EQUIPROBABLE SIGNALS USE ML RULE

$\Rightarrow$ MINIMUM DISTANCE RECEIVER

$$D(\underline{y}, \underline{s}_m) = (\underline{y}-\underline{s}_m)^T(\underline{y}-\underline{s}_m) = \underline{y}^T\underline{y} - 2\underline{y}^T\underline{s}_m + \underline{s}_m^T\underline{s}_m$$

BUT $\underline{s}_m = \sqrt{\mathcal{E}_s}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow m^{th}$ PLACE $= \sqrt{\mathcal{E}_s}\, \underline{e}_m$

$\underline{s}_m^T \underline{s}_m = \mathcal{E}_s$ ALL m

$\Rightarrow$ MAXIMIZE $\underline{y}^T\underline{s}_m = \underline{y}^T\sqrt{\mathcal{E}_s}\,\underline{e}_m = \sqrt{\mathcal{E}_s}\, y_m$

DECIDE $\underline{s}_m$ TRANSMITTED IF y_m IS MAXIMUM.
BY SYMMETRY WE CAN ASSUME $\underline{s}_1$ WAS TRANSMITTED AND

$$P_M = \overset{P(\underline{s}_m)}{\frac{1}{M}} \sum_{m=1}^{M} \underbrace{P(\text{ERROR} \mid \underline{s}_m)}_{\text{WILL BE THE SAME FOR ALL } m}$$

$$= P(\text{ERROR} \mid \underline{s}_1)$$

LET $P_c = 1 - P(\text{ERROR} \mid \underline{S}_1)$

$$= P(y_1 \text{ IS MAXIMUM})$$

$$= \int_{-\infty}^{\infty} P(Y_1 = y_1 \text{ IS MAX.} \mid Y_1 = y_1)\, f_{Y_1}(y_1)\, dy_1$$

$$= \int_{-\infty}^{\infty} P(Y_2 < y_1, Y_3 < y_1, \ldots, Y_M < y_1 \mid_{\wedge} Y_1 = y_1)\, f_{Y_1}(y_1)\, dy_1$$

BUT Y_1 IS INDEPENDENT OF $\{Y_2, \ldots, Y_M\}$

$$= \int_{-\infty}^{\infty} P(Y_2 < y_1, \ldots, Y_M < y_1)\, f_{Y_1}(y_1)\, dy_1$$

AND $Y_2, \ldots, Y_M$ ARE ALL INDEPENDENT AND IDENTICALLY DIST.

$$= \int_{-\infty}^{\infty} P(Y_2 < y_1)^{M-1} f_{Y_1}(y_1)\, dy_1$$

$$= \int_{-\infty}^{\infty} \Big[1 - Q\big(\underbrace{y_1/\sqrt{N_0/2}}_{x}\big)\Big]^{M-1} \frac{1}{\sqrt{\pi N_0}}\, e^{-\frac{1}{N_0}(y_1 - \sqrt{E_s})^2}\, dy_1$$

$$P_M = \int_{-\infty}^{\infty} \left\{1 - [1 - Q(x)]^{M-1}\right\} \frac{1}{\sqrt{\pi N_0}}\, e^{-\frac{1}{N_0}(x\sqrt{N_0/2} - \sqrt{E_s})^2} \sqrt{\tfrac{N_0}{2}}\, dx$$

$$\therefore\ P_M = \int_{-\infty}^{\infty} \left\{1 - [1 - Q(x)]^{M-1}\right\} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(x - \sqrt{2E_s/N_0})^2}\, dx$$

DEPENDS ON M AND $\dfrac{E_s}{N_0/2}$.

TWO MODIFICATIONS ARE

1) EXPRESS IN TERMS OF SNR/BIT $\Rightarrow$ LET $E_s = k E_b$ (SAME METRIC AS PAM)