On the Limitations of Randomization for Queue-Length-Based Scheduling in Wireless Networks

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Abstract—**Randomization is a powerful and pervasive strategy for developing efficient and practical transmission scheduling algorithms in interference-limited wireless networks. Yet, despite the presence of a variety of earlier works on the design and analysis of particular randomized schedulers, there does not exist an extensive study of the limitations of randomization on the efficient scheduling in wireless networks. In this work, we aim to fill this gap by proposing a common modeling framework and three functional forms of randomized schedulers that utilize queue-length information to probabilistically schedule non-conflicting transmissions. This framework not only models many existing schedulers operating under a time-scale separation assumption as special cases, but it also contains a much wider class of potential schedulers that have not been analyzed.**

Our main results are the identification of necessary and sufficient conditions on the network topology and on the functional forms used in the randomization for throughput-optimality. Our analysis reveals an *exponential* and a *sub-exponential* class of functions that exhibit differences in the throughput-optimality. Also, we observe the significance of the network's *scheduling diversity* for throughput-optimality as measured by the number of maximal schedules each link belongs to. We further validate our theoretical results through numerical studies.

I. Introduction

One of the greatest challenges in the efficient communication in wireless networks is the management of interference amongst simultaneous transmissions. A commonly used model, which we also employ in this paper, to capture such interference effects is through the use of a *conflict graph* whereby transmissions that will collide with each other are indicated as conflicting. These conflict graphs can represent a variety of interference models of practical importance, including primary interference model (e.g., [22], [10]), secondary interference model (e.g., [3], [4]), or SINR threshold-based interference model (e.g., [12]). Such conflict graphs can take on extremely complex forms, especially with growing network sizes. Thus, a fundamental question in the design of efficient wireless network protocols is the decision of which subset of non-conflicting transmissions to activate, and when - an operation commonly referred to as *scheduling*.

Of particular interest in the class of scheduling protocols is the set of *throughput-optimal* scheduling strategies (e.g., [26], [17]) that achieves any throughput (subject to network stability) that is achievable by any other scheduling strategy. Thus, throughput-optimal schedulers are critical especially for resource-limited wireless networks as they achieve the largest possible throughput region that is supportable by the network. The seminal works of Tassiulas and Ephremides [26], [27] and many subsequent works (e.g., [5], [17], [24]; see [6] for an overview) have established the throughput-optimality of a variety of *Queue-Length-Based (QLB) Scheduling* strategies, which prioritize activation of links with the greatest backlog awaiting service, also called *Maximum Weight Scheduling (MWS)*.

These original throughput-optimal strategies require the maximum weight schedule to be determined repeatedly as the queue-length levels change. This calls for computationally heavy (even NP-hard in certain interference models) and typically centralized operations, which is impractical. Such restrictions have motivated new research efforts to develop more practical throughput-optimal schedulers with reduced complexity. One such thread led to the development of a class of evolutionary randomized algorithms (also named *pick and compare* algorithms) with throughput-optimality characteristics (see [25], [4], [21]). Another thread led to the development of distributed but suboptimal randomized/greedy strategies (see [14], [9], [2]).

More recently, another exciting thread of results have emerged that can guarantee throughput-optimality by cleverly utilizing queue-length information in the context of carrier sense multiple access (CSMA) (see [15], [8], [19], [18]). In paper [8], the authors proposed an algorithm that adaptively selects the CSMA parameters under a time-scale separation assumption, i.e., the Markov Chain underlying the CSMA-based algorithm converges to steady-state quickly compared with the time-scale of updating parameters of the algorithm. In paper [20], the authors showed the throughput-optimality of a CSMA-based algorithm in which the link weights are chosen to be of the form $\log\log(q+e)$ (where q is the queue length) without the time-scale separation assumption. Ghaderia and Srikant [7] extended these results by showing that the throughput-optimality of CSMA-based algorithm will be preserved even if the link weights have the form $\log(q)/g(q)$, where $g(q)$ can be a function that increases to infinity arbitrarily slowly. Yet, to the best of our knowledge, there does not exist a general framework in which a variety of randomized schedulers can be studied in terms of their throughput-optimality characteristics.

Thus, in this work, we aim to fill this gap by developing a common framework for the modeling and analysis of queue-length-based randomized schedulers, and then by establishing necessary and sufficient conditions on the throughput-optimality of a large functional class of such schedulers under the time-scale separation assumption. Our framework is built

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upon the observation that a common characteristic to most of the developed schedulers is their randomized selection of transmission schedules from the set of all feasible schedules. Specifically, given the existing queue-lengths of the links, each scheduling strategy can be viewed as a particular probability distribution over the set of feasible schedules. While the means with which this random assignment may vary in its distributiveness or complexity, this perspective allows us to model a large set of existing and an even wider set of potential randomized schedulers within a common framework.

This work builds on this original point-of-view to explore the boundaries of randomization in the throughput-optimal operation of wireless networks. Such an investigation is crucial in revealing the necessary and sufficient characteristics of randomized schedulers and the network topologies in which throughput-optimality can be achieved.

Next, we list our main contributions along with references on where they appear in the text.

• In Section II, we introduce three functional forms of randomized queue-length-based scheduling strategies that include many existing strategies as special cases (see Definitions 1, 2 and 3). These strategies differ in the manner in which they measure the weight of schedules, and hence are used to model fundamentally different scheduling implementations.

• We categorize the set of all functions used by these strategies into functions of *exponential form* and of *sub-exponential form* (see Definition 4), collectively covering almost all functions of interest (e.g. $(\log(x+1))^\alpha$, x^α $(\alpha > 0)$ and $\frac{1}{x^\beta}e^{x^\alpha}$ $(\alpha > 0,\ \beta \geq 0)$). These two categories capture the steepness of the functions used in the schedulers, and help reveal a critical degree of steepness necessary for throughput-optimality in large networks.

• Then, we find sufficient (in Section IV) and necessary (in Section V) conditions on the topological characteristics of the conflict graph for the throughput-optimality of these schedulers as a function of the class of functions used in their operation. Our results, graphically summarized in Section III, reveal the significance of the network's scheduling diversity that is measured by the number of schedules each link belongs to.

II. System Model

We consider a fixed wireless network represented by a graph $\mathcal{G} = (\mathcal{N}, \mathcal{L})$, where $\mathcal{N}$ is the set of nodes and $\mathcal{L}$ is the set of undirected links. We assume a time-slotted system, where all nodes transmit at the beginning of each time slot. Due to the interference-limited nature of wireless transmissions, the success or failure of a transmission over a link depends on whether an *interfering* link is also active in the same slot. For ease of exposition, we assume that a successful transmission over any link achieves a unit rate measured in packets per slot.

We use *conflict graphs* to capture any such collision-based interference in the wireless networks. In a *conflict graph* $\mathcal{CG} = (\mathcal{L}, \mathcal{E})$ of $\mathcal{G}$ under a given interference model, the set of links $\mathcal{L}$ in $\mathcal{G}$ becomes the set of nodes, and $\mathcal{E}$ denotes the set of edges that connects links that interfere with each other. In each time slot, we can successfully transmit over nodes in a subset of $\mathcal{L}$ that form an *independent set* (i.e., that are not directly connected in $\mathcal{CG}$). We call each such independent set as a *feasible schedule*, and denote it as $\mathbf{S} = (S_l)_{l\in\mathcal{L}} \in \{0,1\}^{|\mathcal{L}|}$, where $S_l = 1$ if link l is active and $S_l = 0$ is link l is inactive in the schedule. We also treat $\mathbf{S}$ as a set of active links and write $l \in \mathbf{S}$ if $S_l = 1$. We further call a feasible schedule as *maximal* if no more nodes in $\mathcal{CG}$ can be added without violating the interference constraint. As maximal schedules represent extreme points in the space of feasible schedules, we collect them in the set $\mathcal{S}$. Then, we can define the *capacity region* Λ as the convex hull[1] of $\mathcal{S}$ and L-dimensional all-zero vector, which will give the upper bound on the achievable link rates in packets per slot that can be supported by the network under stability for the given interference model.

Given the topology and the interference model of a wireless network, we define the *scheduling diversity of link* $l \in \mathcal{L}$ as the number of different maximal schedules m_l that link l belongs to. Then, for a network topology with a complete N-partite conflict graph[2], we have $\max m_l \leq 1$. As another example, a single-hop wireless network where all links interfere with each other, we have $m_l = 1$ for all l. Less trivially, a 2×2 switch has 2-partite conflict graph in which each maximal schedule has only 2 links, and $m_l = 1$ for each l. Roughly speaking, the scheduling diversity increases as the *network diameter*[3] increases. Such a behavior can be observed directly in a linear network with L links: for $L \leq 3$, $m_l = 1$ for all l; for $L \geq 6$, $m_l \geq 2$ for all l.

In its simplest form, a *scheduler* determines a maximal feasible schedule $\mathbf{S}[t] \in \mathcal{S}$ at each time slot t. This selection may be influenced by the earlier experiences of each transmitter, and may be performed through a variety of strategies. Here, we are not interested in the means of selecting schedules, but in the eventual selection modeled as a probabilistic function of the queue-length state of the network. Before we define the class of randomized schedulers we consider more explicitly, we need to establish the traffic and the queueing models.

For simplicity[4], we assume a per-link traffic model, where $A_l[t]$ arrivals occur to link l in slot t that are independently distributed over links and identically distributed over time with mean λ_l, and $A_l[t] \leq K$ for some $K < \infty$ [5]. Accordingly, a queue is maintained for each link $l \in \mathcal{L}$ with $Q_l[t]$ denoting its queue length at the beginning of time slot t. Recall from above that $S_l[t]$ denotes the number of potential departures at time t. Further, we let $U_l[t]$ denote the unused service for Queue l in slot t. If the queue l is empty and is scheduled, then $U_l[t]$ is equal to 1; otherwise, it is equal to 0. Then, the evolution of

[1] The convex hull of the set $\mathbf{V}$ is the minimal convex set containing set $\mathbf{V}$.

[2] In a complete N-partite conflict graph, the nodes are partitioned into N sets such that every node in each partition is connected to all the nodes of the conflict graph $\mathcal{CG}$ which are not contained in that partition.

[3] Network diameter is the maximum of the shortest hop-count between any two nodes in the graph.

[4] This assumption can be relaxed by utilizing backpressure type routing strategy (see, for example, [26]), which is avoided for unnecessary complications.

[5] We note that the boundedness assumption on the arrival process simplifies the technical arguments, but can be relaxed (see, for example, [5]) to the less strict assumption of $E[A_l^2(t)] < \infty$.

the Queue l is described as follows:

$$Q_l[t+1] = Q_l[t] + A_l[t] - S_l[t] + U_l[t], \qquad \forall l \in \mathcal{L}. \quad (1)$$

We say that Queue l is *f-stable* for a non-negative valued, non-decreasing and divergent function f if it satisfies $\limsup_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}[f(Q_l[t])] < \infty$. We note that this is an extended form of the more traditional strong stability condition (see [6]) that coincide when $f(x) = x$. Moreover, it is easy to show that f-stability implies strong stability when f is also a convex function. We say that the *network is f-stable* if all its queues are f-stable. Accordingly, we say that a scheduler is *f-throughput-optimal* if it achieves f-stability of the network for any arrival rate vector $\lambda = (\lambda_l)_{l\in\mathcal{L}}$ that lies strictly inside the capacity region Λ. Again, in the special case of $f(x) = x$, the notion of f-throughput-optimality reduces to traditional throughput-optimality, and when f is convex, f-throughput-optimality implies throughput-optimality.

Starting with the seminal work [26], there is a vast literature on the design of throughput-optimal schedulers that utilize queue-length information in the selection of the schedules (e.g., [6], [23]). Of special interest in this class of throughput-optimal schedulers are those that employ probabilistic assignments (e.g., [25], [14], [15], [8], [19], [4]). This is not only because they model possible errors in the scheduling process, but also because they allow significant flexibilities in the development of low-complexity and distributed implementations. Yet, randomization causes inaccurate operation and may be hurtful if not performed within limitations.

The aim of this work is to identify the limitations of randomization for a wide class of randomized dynamic schedulers that utilize functions of queue-lengths to schedule transmissions. To that end, we identify three classes of randomized schedulers that differ in the operation of the functional forms used in them. Before we describe them, let us define a basic set of functions we consider:

$\mathcal{F}$:= set of non-negative, nondecreasing and differentiable functions $f(\cdot): \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{x\to\infty} f(x) = \infty$.

Definition 1 (RSOF Scheduler): For a given $f \in \mathcal{F}$ and queue-length vector $\mathbf{Q}$ at the beginning of a slot, the Ratio-of-Sum-of-Functions (RSOF) Scheduler picks a schedule $\mathbf{S} \in \mathcal{S}$ in that slot such that

$$P_{\mathbf{S}}(\mathbf{Q}) := \frac{\sum_{i\in\mathbf{S}} f(Q_i)}{\sum_{\{\mathbf{S}':\mathbf{S}'\in\mathcal{S}\}} \sum_{j\in\mathbf{S}'} f(Q_j)}, \text{ for all } \mathbf{S} \in \mathcal{S} \quad (2)$$

Definition 2 (RMOF Scheduler): For a given $f \in \mathcal{F}$ and queue-length vector $\mathbf{Q}$ at the beginning of a slot, the Ratio-of-Multiplication-of-Functions (RMOF) Scheduler picks a schedule $\mathbf{S} \in \mathcal{S}$ in that slot such that

$$\upsilon_{\mathbf{S}}(\mathbf{Q}) := \frac{\prod_{i\in\mathbf{S}} f(Q_i)}{\sum_{\{\mathbf{S}':\mathbf{S}'\in\mathcal{S}\}} \prod_{j\in\mathbf{S}'} f(Q_j)}, \text{ for all } \mathbf{S} \in \mathcal{S} \quad (3)$$

Definition 3 (RFOS Scheduler): For a given $f \in \mathcal{F}$ and queue-length vector $\mathbf{Q}$ at the beginning of a slot, the Ratio-of-Function-of-Sums (RFOS) Scheduler picks a schedule $\mathbf{S} \in \mathcal{S}$ in that slot such that

$$\pi_{\mathbf{S}}(\mathbf{Q}) := \frac{f(\sum_{i\in\mathbf{S}} Q_i)}{\sum_{\{\mathbf{S}':\mathbf{S}'\in\mathcal{S}\}} f(\sum_{j\in\mathbf{S}'} Q_j)}, \text{ for all } \mathbf{S} \in \mathcal{S} \quad (4)$$

Note that all the RSOF, RMOF and RFOS Schedulers are more likely to pick a schedule with the larger queue length, but with different distributions based on their form and the form of $f \in \mathcal{F}$. In particular, the steepness of the function f determines the weight given to the heavily loaded *link* in both RSOF and RMOF Schedulers and the heavily loaded *schedule* in the RFOS Scheduler. Also, note that the schedulers coincide in single-hop network topologies and for the following choices of f: when $f(x) = x$, RSOF and RFOS Schedulers coincide; when $f(x) = e^x$, RMOF and RFOS Schedulers coincide. These three classes cover a wide variety of schedulers including many of existing throughput-optimal schedulers. For example, when $f(x) = e^x$, the RMOS and RFOS Schedulers correspond to the throughput-optimal CSMA policy operating under time-scale separation assumption that attracted a lot of attention lately (see [8], [19], [18]). Yet, they also contain a much wider set of schedulers, one for each f.

It is important to understand the variety of functional forms that may achieve throughput-optimality since they are likely to possess differences in their implementation complexity and distributiveness characteristics. For example, for a given distribution, we can construct a Markov Chain that converges to it by Metropolis algorithm [16]. Moreover, the RMOF scheduler can be implemented distributively through the Glauber dynamics (e.g., [20], [7]).

Next, we identify the three classes of functions with varying forms and steepness that turn out to be crucial to our investigation.

Definition 4: We consider the following subsets of the set of functions $\mathcal{F}$:

(a) $\mathcal{A} := \{f \in \mathcal{F} : \forall\epsilon > 0, \lim_{x\to\infty} \frac{f(x)}{f((1+\epsilon)x)} = 0\}$. We call $\mathcal{A}$ as the *class of exponential functions*.

(b) $\mathcal{B} := \{f \in \mathcal{F}$: $\lim_{x\to\infty} \frac{f(x+a)}{f(x)} = 1$, for any $a \in \mathbb{R}\}$.

(c) $\mathcal{C} := \{f \in \mathcal{B}$: there exist K_1 and K_2 satisfying $0 < K_1 \le K_2 < \infty$ such that $K_1(f(x_1) + f(x_2)) \le f(x_1 + x_2) \le K_2(f(x_1) + f(x_2))$, for all $x_1, x_2 \ge 0\}$.
We call $\mathcal{C}$ as the *class of sub-exponential functions*.

The key examples of functions with sets A, B, C and their interrelationship are extensively studied in our technical report [13]. Instead, Figure 1 concisely demonstrates the most critical facts: that $\mathcal{A}$ and $\mathcal{C}$ are non-overlapping classes; while $\mathcal{B}$ has an intersection with $\mathcal{A}$. Furthermore, the example functions are provided with a variety of forms that justify the names assigned to $\mathcal{A}$ and $\mathcal{C}$: the set $\mathcal{A}$ contains rapidly increasing functions generally with exponential forms; while the set $\mathcal{C}$ contains sub-exponentially increasing polynomial and logarithmic functional forms. In the study of necessary and sufficient conditions for throughput-optimality, we shall find that most of the results depend on which of these three functional classes the functions belong to.

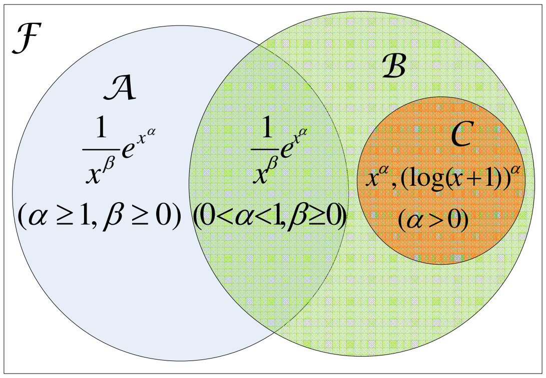


Fig. 1: The content of and the relationship between classes $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.

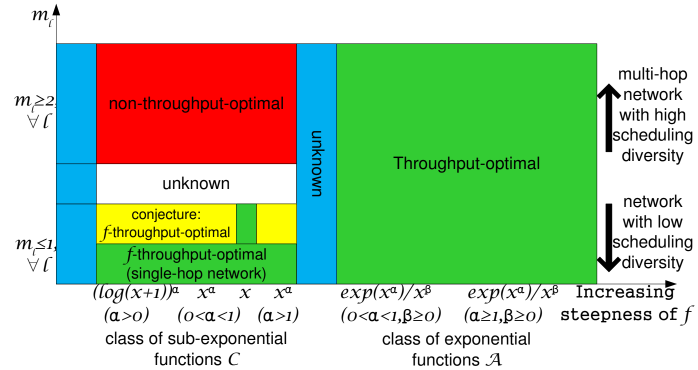


Fig. 3: The throughput performance of the RFOS Scheduler.

III. Overview of Main Results

In this section, we present our main findings and resulting insights on the throughput-optimality of the RSOF, RMOF and RFOS Schedulers (see Definitions 1, 2 and 3) with different functional forms under different network topologies. These results are rigorously proven in Sections IV and V. To facilitate an accessible figurative presentation, in the horizontal dimension, we conceptually order functions in $\mathcal{F}$ in increasing level of steepness starting from $f(x) = (\log(x+1))^\alpha$ and $f(x) = x^\alpha$ for any $\alpha > 0$ that belong to $\mathcal{C}$, followed by $f(x) = \frac{1}{x^\beta}e^{x^\alpha}$ for any $0 < \alpha < 1$ and any $\beta \geq 0$ that belongs to $\mathcal{B} \bigcap \mathcal{A}$, and finishing with $f(x) = \frac{1}{x^\beta}e^{x^\alpha}$ for any $\alpha \geq 1$ and any $\beta \geq 0$ that belongs to $\mathcal{A}$. In the vertical dimension, we use the scheduling diversity $(m_l)_{l\in\mathcal{L}}$ introduced in Section II to distinguish different topological and interference scenarios. Recall that since m_l denotes the number of different maximal schedules that link l belongs to, it is a *rough* measure of the multi-hop nature of the network. Then, the main results for RSOF and RFOS Schedulers are presented in Figures 2 and 3, respectively. In these figures, besides proven results, we also include several conjectures that are validated through simulations in Section VI.

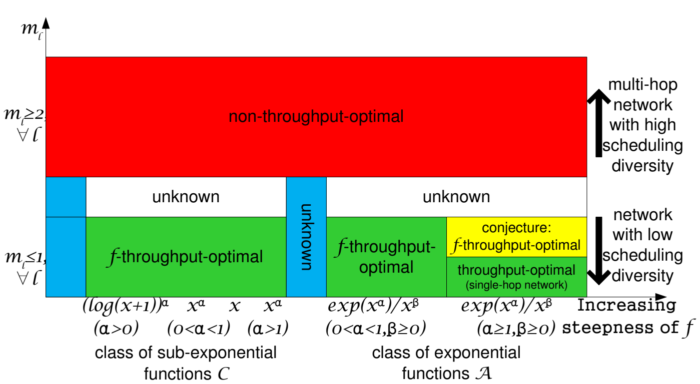


Fig. 2: The throughput performance of the RSOF Scheduler.

From Figure 2, we see that the RSOF Scheduler with the function $f \in \mathcal{B}$ is f-throughput-optimal when $\max_{l\in\mathcal{L}} m_l \leq 1$. Also, the RSOF Scheduler with the function $f \in A \setminus B$ is throughput-optimal in single-hop networks since RSOF and RFOS Schedulers have the same probability distribution over schedules in such networks. However, if $\min_{l\in\mathcal{L}} m_l \geq 2$, the RSOF Scheduler with any function $f \in \mathcal{F}$ cannot be throughput-optimal. Thus, roughly speaking, the RSOF Scheduler is non-throughput-optimal for the network with high scheduling diversity, while the RSOF Scheduler with the function $f \in \mathcal{B}$ is f-throughput-optimal for low scheduling diversity. We note that although the throughput performance of RSOF Scheduler with some exponential functions $f \in A \setminus B$ (i.e. $f(x) = \frac{1}{x^\beta}e^{x^\alpha}$, $\alpha \geq 1$ and $\beta \geq 0$) is not yet explored in general topologies with $\max_{l\in\mathcal{L}} m_l \leq 1$, we conjecture that it is f-throughput-optimal in this region, since the RSOF Scheduler with such functions reacts much more quickly to the queue length difference between schedules than that with sub-exponential functions, especially under asymmetric arrival patterns. We validate this conjecture through simulations in Section VI.

The horizontal unknown region corresponds to network topologies where some links have scheduling diversity 1 and other links have scheduling diversity at least 2. The vertical unknown region corresponds to randomized schedulers with functions f that are not in the functional classes $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. In Figure 3, we observe that the RFOS Scheduler with the function $f \in \mathcal{A}$ is throughput-optimal under any network topology. Also, the RFOS Scheduler with the function $f \in \mathcal{C}$ is f-throughput-optimal in single-hop networks where RFOS and RSOF Schedulers have the same probability distribution over schedulers. Also, when the function f is linear, the RFOS Scheduler has the same probability form with the RSOF Scheduler and thus is f-throughput-optimal when $\max_{l\in\mathcal{L}} m_l \leq 1$. However, the RFOS Scheduler with the function $f \in \mathcal{C}$ is not throughput-optimal when $\min_{l\in\mathcal{L}} m_l \geq 2$. Roughly speaking, the network with higher scheduling diversity requires much steeper functions (e.g., exponential functions) for the throughput-optimality of the RFOS Scheduler. While the throughput performance of RFOS Scheduler with the function $f \in \mathcal{C} \setminus \{\text{linear functions}\}$ for general network topologies with $\max_{l\in\mathcal{L}} m_l \leq 1$ is part of our ongoing work, we conjecture that it is f-throughput-optimal in those topologies since both RFOS and RSOF Schedulers with sub-exponential functions have almost the same reaction speed to the queue length difference between schedules. We also validate this conjecture via simulations in Section VI.

The RMOF Scheduler with the function f satisfying $\log f \in \mathcal{B}$ and $f(0) \geq 1$ is $(\log f)$-throughput-optimal under any network topology. This result together with the RFOS Scheduler with the function $f \in \mathcal{A}$ extends the throughput-optimality of CSMA schedulers (e.g. [8], [18]) to a wider class of functional forms. While this result proves a weaker form of throughput-optimality than f-throughput-optimality for the RMOF Scheduler, we note that RMOF Scheduler generally outperforms RFOS and RSOF Schedulers in our numerical investigations. Hence, we leave it to future research to strengthen this result.

Collectively these results not only highlight the strengths and weaknesses of the three functional randomized schedulers, they also reveal the interrelation between the steepness of the functions and the scheduling diversity of the underlying wireless networks. This extensive understanding of the limitations of randomization may motivate the network designers to use or avoid certain types of probabilistic scheduling strategies depending on the topological characteristics of the network.

IV. Sufficient Conditions

In this section, we study the sufficient conditions on the network's topological characteristics and the functions used in RSOF, RMOF and RFOS Schedulers to achieve throughput-optimality.

A. f-Throughput-Optimality of RSOF Scheduler

We study the throughput performance of the RSOF Scheduler for a network topology with $\max_{l\in\mathcal{L}} m_l \leq 1$. In such a network, each link only belongs to one maximal schedule.

Lemma 1: If $\sum_{i=1}^N \lambda_i < 1$, $\lambda_i > 0$ and $a_i \geq 0$, for $i = 1, ..., N$, then there exists a $\delta > 0$ such that

$$\sum_{i=1}^N \frac{a_i^2}{\lambda_i} > (\sum_{i=1}^N a_i)^2(1+\delta) \tag{5}$$

Proof: See our technical report [13] for the proof. ■

Theorem 1: In a network topology with $\max_{l\in\mathcal{L}} m_l \leq 1$, the RSOF Scheduler with the function $f \in \mathcal{B}$ is f-throughput-optimal.

Proof: Without loss of generality, we assume that there are only N available maximal schedules $\mathbf{S}^i$ $(i = 1, ..., N)$. Since each link belongs to one maximal schedule, we can denote the queues, arrivals, and scheduling statistics in terms of maximal schedules for easier exposition. To that end, we let Q_l^i, λ_l^i and P_l^i $(i = 1, ..., N, l = 1, ..., |\mathbf{S}^i|)$ denote the queue-length of link $l \in \mathbf{S}^i$, the average arrival rate for the link $l \in \mathbf{S}^i$ and probability of serving the link $l \in \mathbf{S}^i$, respectively. In addition, $A_l^i[t]$, $S_l^i[t]$ and $U_l^i[t]$ denote the number of arrivals to link $l \in \mathbf{S}^i$ at time slot t, the number of potential departures of link $l \in \mathbf{S}^i$ in slot t and the unused service for link $l \in \mathbf{S}^i$ at time slot t, respectively. The capacity region for such network is

$$C_N := \{\lambda : \sum_{i=1}^N \lambda_{l_i}^i < 1, \forall i = 1, ...N, l_i = 1, ..., |\mathbf{S}^i|\} \tag{6}$$

Under the above notation, the RSOF Scheduler becomes :

$$P_{\mathbf{S}^i} = \frac{\sum_{l=1}^{|\mathbf{S}^i|} f(Q_l^i)}{\sum_{k=1}^N \sum_{l=1}^{|\mathbf{S}^k|} f(Q_l^k)}, i = 1, ..., N. \tag{7}$$

Note that $P_l^i = P_{\mathbf{S}^i}$, for $i = 1, ..., N$, $l = 1, ..., |\mathbf{S}^i|$. If $\lambda_l^i = 0$ for some i and l, then no arrivals occur in the link $l \in \mathbf{S}^i$. Thus, we don't need to consider such links. Follows we assume $\lambda_l^i > 0$ $(i = 1, ..., N,\ l = 1, ..., |\mathbf{S}^i|)$. Consider the Lyapunov function $V(\mathbf{Q}) := \sum_{i=1}^N \sum_{l=1}^{|\mathbf{S}^i|} \frac{h(Q_l^i)}{\lambda_l^i}$, where $h'(x) = f(x)$. Then

$$\begin{aligned}\Delta V :&= \mathbb{E}\left[V(\mathbf{Q}[\mathbf{t+1}]) - V(\mathbf{Q}[\mathbf{t}]) | \mathbf{Q}[t] = \mathbf{Q}\right] \\ &= \sum_{i=1}^N \sum_{l=1}^{|\mathbf{S}^i|} \mathbb{E}\left[\frac{1}{\lambda_l^i}(h(Q_l^i[t+1]) - h(Q_l^i[t]))|\mathbf{Q}[t] = \mathbf{Q}\right]\end{aligned}$$

By the mean-value theorem, we have $h(Q_l^i[t+1]) - h(Q_l^i[t]) = f(R_l^i[t])(Q_l^i[t+1] - Q_l^i[t]) = f(R_l^i[t])(A_l^i[t] - S_l^i[t] + U_l^i[t])$, where $R_l^i[t]$ lies between $Q_l^i[t]$ and $Q_l^i[t+1]$. Hence, we get

$$\begin{aligned}\Delta V &= \sum_{i=1}^N \sum_{l=1}^{|\mathbf{S}^i|} \mathbb{E}\left[\frac{1}{\lambda_l^i} f(R_l^i[t])(A_l^l[t] - S_l^i[t] + U_l^i[t])|\mathbf{Q}[t] = \mathbf{Q}\right] \\ &= \underbrace{\sum_{i=1}^N \sum_{l=1}^{|\mathbf{S}^i|} \mathbb{E}\left[\frac{1}{\lambda_l^i} f(R_l^i[t]) U_l^i[t] |\mathbf{Q}[t] = \mathbf{Q}\right]}_{=:\Delta V_1} + \\ &\quad \underbrace{\sum_{i=1}^N \sum_{l=1}^{|\mathbf{S}^i|} \mathbb{E}\left[\frac{1}{\lambda_l^i} f(R_l^i[t])(A_l^i[t] - S_l^i[t])|\mathbf{Q}[t] = \mathbf{Q}\right]}_{=:\Delta V_2}\end{aligned}$$

For ΔV_1, if $Q_l^i[t] = Q_l^i > 0$, then $U_l^i[t] = 0$. If $Q_l^i[t] = Q_l^i = 0$, then $U_l^i[t]$ may be equal to 1. But in this case, $Q_l^i[t+1] \leq K$ (since $A_l^i[t] \leq K$). Hence, $f(R_l^i[t]) \leq f(K) < \infty$. Thus,

$$\begin{aligned}\Delta V_1 &= \sum_{i=1}^N \sum_{l=1}^{|\mathbf{S}^i|} \mathbb{E}\left[\frac{1}{\lambda_l^i} f(R_l^i[t]) U_l^i[t] |\mathbf{Q}[t] = \mathbf{Q}\right] \mathbf{1}_{\{Q_l^i = 0\}} \\ &\leq \sum_{i=1}^N \sum_{l=1}^{|\mathbf{S}^i|} \frac{1}{\lambda_l^i} f(K) \leq D \sum_{i=1}^N \sum_{l=1}^{|\mathbf{S}^i|} f(K)\end{aligned} \tag{8}$$

Where $D := \frac{1}{\min\{\lambda_l^i\}} < \infty$ and $\mathbf{1}_{\{.\}}$ is the indicator function.

Next, let's focus on ΔV_2. We know that $f(R_l^i[t]) = f(Q_l^i[t] + a_l^i)$ $(|a_l^i| \leq K)$. According to the definition of function $f \in \mathcal{B}$, given $\epsilon > 0$, there exists $M > 0$, such that for any $Q_l^i[t] = Q_l^i > M$, we have $\left|\frac{f(R_l^i[t])}{f(Q_l^i)} - 1\right| < \epsilon$, that is,

$$(1-\epsilon) f(Q_l^i) < f(R_l^i[t]) < (1+\epsilon) f(Q_l^i) \tag{9}$$

Thus, we have

$$\begin{aligned}&f(R_l^i[t])(A_l^i[t]-S_l^i[t])\\=&f(R_l^i[t])\left[(A_l^i[t]-S_l^i[t])_+-(A_l^i[t]-S_l^i[t])_-\right]\\<&(1+\epsilon)f(Q_l^i)(A_l^i[t]-S_l^i[t])_+-(1-\epsilon)f(Q_l^i)(A_l^i[t]-S_l^i[t])_-\\=&f(Q_l^i)(A_l^i[t]-S_l^i[t])+\epsilon f(Q_l^i)\left|A_l^i[t]-S_l^i[t]\right|\\\leq&f(Q_l^i)(A_l^i[t]-S_l^i[t])+K\epsilon f(Q_l^i)\end{aligned}\tag{10}$$

Where $(x)_+=\max\{x,0\}$, $(x)_-=-\min\{x,0\}$ and $|A_l^i[t]-S_l^i[t]|\leq|A_l^i[t]|\leq K$. Thus, we divide ΔV_2 into two parts:

$$\Delta V_2=\underbrace{\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\mathbb{E}\left[\frac{1}{\lambda_l^i}f(R_l^i[t])(A_l^i[t]-S_l^i[t])|\mathbf{Q}[t]=\mathbf{Q}\right]\mathbf{1}_{\{Q_l^i>M\}}}_{=:\Delta V_3}$$
$$+\underbrace{\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\mathbb{E}\left[\frac{1}{\lambda_l^i}f(R_l^i[t])(A_l^i[t]-S_l^i[t])|\mathbf{Q}[t]=\mathbf{Q}\right]\mathbf{1}_{\{Q_l^i\leq M\}}}_{=:\Delta V_4}$$

For ΔV_3, by using (10), we have

$$\begin{aligned}\Delta V_3\leq&\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\mathbb{E}\left[\frac{1}{\lambda_l^i}f(Q_l^i)(A_l^i[t]-S_l^i[t])|\mathbf{Q}[t]=\mathbf{Q}\right]\mathbf{1}_{\{Q_l^i>M\}}\\&+\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\frac{1}{\lambda_l^i}K\epsilon f(Q_l^i)\mathbf{1}_{\{Q_l^i>M\}}\\\leq&\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\frac{1}{\lambda_l^i}f(Q_l^i)(\lambda_l^i-P_l^i)\mathbf{1}_{\{Q_l^i>M\}}+DK\epsilon\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i)\mathbf{1}_{\{Q_l^i>M\}}\end{aligned}$$

where $P_l^i=\mathbb{E}\left[S_l^i[t]|\mathbf{Q}[t]=\mathbf{Q}\right]=\frac{\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i)}{\sum_{k=1}^{N}\sum_{l=1}^{|\mathbf{S}^k|}f(Q_l^k)}$. Consider the term $\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\frac{1}{\lambda_l^i}f(Q_l^i)(\lambda_l^i-P_l^i)$,

$$\begin{aligned}&\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\frac{1}{\lambda_l^i}f(Q_l^i)(\lambda_l^i-P_l^i)\\=&\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i)-\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\frac{f(Q_l^i)}{\lambda_l^i}\frac{\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i)}{\sum_{k=1}^{N}\sum_{l=1}^{|\mathbf{S}^k|}f(Q_l^k)}\\=&\frac{(\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i))^2-\sum_{i=1}^{N}(\sum_{l=1}^{|\mathbf{S}^i|}\frac{f(Q_l^i)}{\lambda_l^i})(\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i))}{\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i)}\end{aligned}$$

Since

$$\sum_{i=1}^{N}(\sum_{l=1}^{|\mathbf{S}^i|}\frac{f(Q_l^i)}{\lambda_l^i})(\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i))>\sum_{i=1}^{N}\frac{1}{\lambda^i}(\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i))^2$$

where $\lambda^i=\max_{\{l=1,\ldots,|\mathbf{S}^i|\}}\lambda_l^i$, and by Lemma 1, there exists a $\delta>0$ such that

$$\sum_{i=1}^{N}\frac{1}{\lambda^i}(\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i))^2>(\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i))^2(1+\delta)\tag{11}$$

we have

$$\sum_{i=1}^{N}(\sum_{l=1}^{|\mathbf{S}^i|}\frac{f(Q_l^i)}{\lambda_l^i})(\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i))>(\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i))^2(1+\delta)$$

Thus, we get

$$\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\frac{1}{\lambda_l^i}f(Q_l^i)(\lambda_l^i-P_l^i)<-\delta\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i)\tag{12}$$

Hence,

$$\begin{aligned}&\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\frac{1}{\lambda_l^i}f(Q_l^i)(\lambda_l^i-P_l^i)\mathbf{1}_{\{Q_l^i>M\}}\\<&-\delta\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i)\mathbf{1}_{\{Q_l^i>M\}}-\delta\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i)\mathbf{1}_{\{Q_l^i\leq M\}}-\\&\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\frac{1}{\lambda_l}f(Q_l^i)(\lambda_l^i-P_l^i)\mathbf{1}_{\{Q_l^i\leq M\}}\\<&-\delta\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i)\mathbf{1}_{\{Q_l^i>M\}}+\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\frac{1}{\lambda_l^i}f(Q_l^i)P_l^i\mathbf{1}_{\{Q_l^i\leq M\}}\\\leq&-\delta\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i)\mathbf{1}_{\{Q_l^i>M\}}+D\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(M)\end{aligned}\tag{13}$$

Thus, we can choose ϵ small enough such that $\gamma=\delta-DK\epsilon>0$ and thus we have

$$\Delta V_3<-\gamma\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i)\mathbf{1}_{\{Q_l^i>M\}}+D\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(M)$$

For ΔV_4, we have

$$\begin{aligned}\Delta V_4\leq&\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\mathbb{E}\left[\frac{1}{\lambda_l^i}f(R_l^i[t])|A_l^i[t]-S_l^i[t]||\mathbf{Q}[t]=\mathbf{Q}\right]\mathbf{1}_{\{Q_l^i\leq M\}}\\\leq&\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\frac{1}{\lambda_l^i}Kf(M+K)\leq DK\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(M+K)\end{aligned}$$

Thus, we get

$$\Delta V<-\gamma\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i)\mathbf{1}_{\{Q_l^i>M\}}+C\tag{14}$$

where $C:=D\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(M)+DK\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(M+K)+D\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(K)<\infty$. By Foster-Lyapunov theorem [1], equation (14) implies f-throughput-optimality of the RSOF Scheduler for any $f\in\mathcal{B}$. ■

B. Throughput-Optimality of RMOF and RFOS Schedulers

In this subsection, we investigate the sufficient condition for the throughput-optimality of RMOF and RFOS Schedulers.

Theorem 2: (i) The RMOF Scheduler with the function $f\in\mathcal{F}$ satisfying $\log f\in\mathcal{B}$ and $f(0)\geq 1$ is $(\log f)$-throughput-optimal under any network topology;
(ii) The RFOS Scheduler with the function $f\in\mathcal{A}$ is throughput-optimal under any network topology.

Proof: To prove this, we use a similar approach as in [18] that uses the following result from [5]: for a scheduling algorithm, if given any $0\leq\epsilon,\delta<1$, there exists a $M>0$ such that the scheduling algorithm satisfies the following condition: in any time slot t, with probability greater than $1-\delta$, the scheduling algorithm chooses a schedule $\mathbf{x}(t) \in \mathcal{S}$ that satisfies:

$$\sum_{l \in \mathbf{x}(t)} w(Q_l(t)) \geq (1-\epsilon) \max_{\mathbf{x} \in \mathcal{S}} \sum_{l \in \mathbf{x}(t)} w(Q_l(t)) \quad (15)$$

whenever $\| \mathbf{Q(t)} \| > M$, where $\mathbf{Q(t)} := (Q_l(t) : l \in \mathcal{L})$ and $w \in \mathcal{B}$. Then the scheduling algorithm is w-throughput-optimal.
(i) Given any ϵ and δ such that $0 \leq \epsilon, \delta < 1$. Let

$$\mathcal{X} := \{x \in \mathcal{S} : \sum_{l \in \mathbf{x}} \log f(Q_l(t)) < (1-\epsilon) W^*\} \quad (16)$$

Where $W^* := \max_{\mathbf{x} \in \mathcal{S}} \sum_{l \in \mathbf{x}} \log f(Q_l(t))$ and $\mathbf{x}^* := \arg\max_{\mathbf{x} \in \mathcal{S}} \sum_{l \in \mathbf{x}} \log f(Q_l(t))$. Then

$$\upsilon(\mathcal{X}) = \sum_{\mathbf{x} \in \mathcal{X}} \upsilon_{\mathbf{x}} = \sum_{\mathbf{x} \in \mathcal{X}} \frac{\prod_{l \in \mathbf{x}} f(Q_l(t))}{\sum_{\mathbf{x}' \in \mathcal{S}} \prod_{l \in \mathbf{x}'} f(Q_l(t))}$$
$$= \frac{\sum_{\mathbf{x} \in \mathcal{X}} \exp\left[\sum_{l \in \mathbf{x}} \log f(Q_l(t))\right]}{\sum_{\mathbf{x} \in \mathcal{S}} \exp\left[\sum_{l \in \mathbf{x}} \log f(Q_l(t))\right]} < \frac{|\mathcal{X}| \exp\left[(1-\epsilon) W^*\right]}{\sum_{\mathbf{x} \in \mathcal{S}} \exp\left[\sum_{l \in \mathbf{x}} \log f(Q_l(t))\right]}$$

Since $\sum_{\mathbf{x} \in \mathcal{S}} \exp\left[\sum_{l \in \mathbf{x}} \log f(Q_l(t))\right] \geq \exp(W^*)$, then we have

$$\upsilon(\mathcal{X}) < \frac{|\mathcal{X}| \exp\left[(1-\epsilon) W^*\right]}{\exp(W^*)} = \frac{|\mathcal{X}|}{\exp(\epsilon W^*)} \quad (17)$$

Thus if some queue lengths increase to infinity, then $W^* \to \infty$ and thus we have $\upsilon(\mathcal{X}) \to 0$. Hence the RMOF Scheduler with the function $f \in \mathcal{F}$ satisfying $\log f \in \mathcal{B}$ and $f(0) \geq 1$ is $\log f$-throughput-optimal under any topology.
(ii) Given any ϵ and δ such that $0 \leq \epsilon, \delta < 1$. Let $Q^*(t) := \max_{\mathbf{x} \in \mathcal{S}} \sum_{l \in \mathbf{x}} Q_l(t)$ and $\mathbf{x}^* := \arg\max_{\mathbf{x} \in \mathcal{S}} \sum_{l \in \mathbf{x}} Q_l(t)$. Let $\mathcal{X} := \{x \in \mathcal{S} : \sum_{l \in \mathbf{x}} Q_l(t) < (1-\epsilon) Q^*(t)\}$. Then, using the same technique as in (i), we can prove that RFOS Scheduler with the function $f \in \mathcal{A}$ is throughput-optimal under any topology. ∎

V. Necessary Conditions

So far, we have shown that the RSOF Scheduler with the function $f \in \mathcal{B}$ is f-throughput-optimal in the network topology with $\max_{l \in \mathcal{L}} m_l \leq 1$, the RMOF Scheduler with the function $f \in \mathcal{F}$ satisfying $\log f \in B$ and $f(0) \geq 1$ is $(\log f)$-throughput-optimal in general network topologies and the RFOS Scheduler with the function $f \in \mathcal{A}$ is throughput-optimal under arbitrary network topologies. However, the next result establishes that in network topologies where each link belongs to two or more schedules (i.e. when $\min_{l \in \mathcal{L}} m_l \geq 2$), the RSOF Scheduler with any function $f \in \mathcal{F}$ and RFOS Scheduler with the function $f \in \mathcal{C}$ cannot be throughput-optimal.

Theorem 3: If the network is such that $\min_{l \in \mathcal{L}} m_l \geq 2$, then (i) RSOF Scheduler is not throughput-optimal for any $f \in \mathcal{F}$; (ii) RFOS Scheduler is not throughput-optimal for any $f \in \mathcal{C}$.

Proof: We prove these claims constructively by considering an arrival process that is inside the capacity region, but is not supportable by the randomized schedulers for the given functional forms. To that end, let us consider any maximal schedule $\mathbf{S}_0 \in \mathcal{S}$ and index its links as $\{1, 2, ..., n\}$ for convenience. We assume that arrivals only happen to those n links at rates $\lambda_1, \cdots, \lambda_n$ with the constraint that $\lambda_i \in [0, 1)$ for all $i = 1, \cdots, n$, which is clearly supportable by a simple scheduling policy that always serves the schedule $\mathbf{S}_0$. Thus, setting λ_i arbitrarily close to one for each i, this simple policy can achieve a sum rate of $\sum_{i=1}^n \lambda_i < n$.

We define $\mathcal{M} = \{\mathbf{S} \in \mathcal{S} : \mathbf{S} \bigcap \mathbf{S_0} \neq \emptyset\}$, $\mathcal{K} = \mathcal{S} \setminus \mathcal{M}$, $\mathcal{H} = \mathcal{M} \setminus \{\mathbf{S_0}\}$ and $\mathcal{T} = \mathcal{S} \setminus \{\mathbf{S_0}\}$. In the rest of the proof, we use $|\mathbf{A}|$ to denote the cardinality of the set $\mathbf{A}$ and $\mathbf{AB}$ to denote the intersection of $\mathbf{A}$ and $\mathbf{B}$.

Given this construction, we next prove the following statements for the RSOF and RFOS Schedulers respectively:
(1) If $\sum_{i=1}^n \lambda_i \geq n - \frac{1}{2}$, the RSOF Scheduler with any function $f \in \mathcal{F}$ is unstable.
(2) If $\sum_{i=1}^n \lambda_i \geq n - \frac{K_1}{2K_2}$, where $K_1 \leq K_2$ are positive constants described in Definition 4(c), the RFOS Scheduler with the associated function $f \in \mathcal{C}$ is unstable.

Since the aforementioned simple scheduler can stabilize the sum rate $\sum_{i=1}^n \lambda_i < n$, the RSOF Scheduler with any function $f \in \mathcal{F}$ and RFOS Scheduler with the associated function $f \in \mathcal{C}$ are not throughput-optimal. We next prove these claims that complete the proof of Theorem 3.
(1) Under the above model, the RSOF Scheduler becomes

$$P_{\mathbf{S}} = \frac{\sum_{l \in \mathbf{SS_0}} f(Q_l) + |\mathbf{S} \setminus \mathbf{S_0}| f(0)}{\sum_{\mathbf{S}':\mathbf{S}' \in \mathcal{S}} (\sum_{l \in \mathbf{S}'\mathbf{S_0}} f(Q_l) + |\mathbf{S}' \setminus \mathbf{S_0}| f(0))}$$

Let P_l denote the probability that link $l \in \mathbf{S}_0$ is served, then

$$\sum_{l=1}^n P_l = \sum_{l=1}^n \sum_{\mathbf{S} \in \mathcal{M} : l \in \mathbf{SS_0}} P_{\mathbf{S}}$$
$$= \frac{\sum_{l=1}^n \sum_{\mathbf{S} \in \mathcal{M} : l \in \mathbf{SS_0}} (\sum_{i \in \mathbf{SS_0}} f(Q_i) + |\mathbf{S} \setminus \mathbf{S_0}| f(0))}{\sum_{\mathbf{S}:\mathbf{S} \in \mathcal{S}} \sum_{l \in \mathbf{SS_0}} f(Q_l) + \sum_{\mathbf{S}:\mathbf{S} \in \mathcal{S}} |\mathbf{S} \setminus \mathbf{S_0}| f(0)}$$

Since $\sum_{\mathbf{S}:\mathbf{S} \in \mathcal{S}} \sum_{l \in \mathbf{SS_0}} f(Q_l) = \sum_{l=1}^n f(Q_l) \sum_{\mathbf{S} \in \mathcal{M} : l \in \mathbf{SS_0}} 1$, $\sum_{l=1}^n \sum_{\mathbf{S} \in \mathcal{M} : l \in \mathbf{SS_0}} |\mathbf{S} \setminus \mathbf{S_0}| f(0) = \sum_{\mathbf{S}:\mathbf{S} \in \mathcal{S}} |\mathbf{SS_0}| |\mathbf{S} \setminus \mathbf{S_0}| f(0)$ and

$$\sum_{l=1}^n \sum_{\mathbf{S} \in \mathcal{M} : l \in \mathbf{SS_0}} \sum_{i \in \mathbf{SS_0}} f(Q_i) = \sum_{\mathbf{S}:\mathbf{S} \in \mathcal{M}} \sum_{l \in \mathbf{SS_0}} \sum_{i \in \mathbf{SS_0}} f(Q_i)$$
$$= \sum_{\mathbf{S}:\mathbf{S} \in \mathcal{M}} |\mathbf{SS_0}| \sum_{i \in \mathbf{SS_0}} f(Q_i) = \sum_{l=1}^n f(Q_l) \sum_{\mathbf{S} \in \mathcal{M} : l \in \mathbf{SS_0}} |\mathbf{SS_0}|$$

we can extend the above ratio as follows:

$$\sum_{l=1}^n P_l = \frac{\sum_{l=1}^n f(Q_l) \sum_{\mathbf{S} \in \mathcal{M} : l \in \mathbf{SS_0}} |\mathbf{SS_0}| + \sum_{\mathbf{S}:\mathbf{S} \in \mathcal{S}} |\mathbf{SS_0}| |\mathbf{S} \setminus \mathbf{S_0}| f(0)}{\sum_{l=1}^n f(Q_l) \sum_{\mathbf{S} \in \mathcal{M} : l \in \mathbf{SS_0}} 1 + \sum_{\mathbf{S}:\mathbf{S} \in \mathcal{S}} |\mathbf{S} \setminus \mathbf{S_0}| f(0)}$$
$$= \frac{\sum_{l=1}^n f(Q_l)(n + \sum_{\mathbf{H} \in \mathcal{H} : l \in \mathbf{HS_0}} |\mathbf{HS_0}|) + \sum_{\mathbf{T}:\mathbf{T} \in \mathcal{T}} |\mathbf{TS_0}| |\mathbf{T} \setminus \mathbf{S_0}| f(0)}{\sum_{l=1}^n f(Q_l)(1 + \sum_{\mathbf{H} \in \mathcal{H} : l \in \mathbf{HS_0}} 1) + \sum_{\mathbf{T}:\mathbf{T} \in \mathcal{T}} |\mathbf{T} \setminus \mathbf{S_0}| f(0)}$$
$$= n - \frac{\sum_{l=1}^n f(Q_l) \sum_{\mathbf{H} \in \mathcal{H} : l \in \mathbf{HS_0}} (n - |\mathbf{HS_0}|) + \sum_{\mathbf{T}:\mathbf{T} \in \mathcal{T}} (n - |\mathbf{TS_0}|) |\mathbf{T} \setminus \mathbf{S_0}| f(0)}{\sum_{l=1}^n f(Q_l)(1 + \sum_{\mathbf{H} \in \mathcal{H} : l \in \mathbf{HS_0}} 1) + \sum_{\mathbf{T}:\mathbf{T} \in \mathcal{T}} |\mathbf{T} \setminus \mathbf{S_0}| f(0)}$$

Note that $|\mathbf{HS_0}| \leq n-1$, for $\forall \mathbf{H} \in \mathcal{H}$ and $|\mathbf{TS_0}| \leq n-1$, for $\forall \mathbf{T} \in \mathcal{T}$. Now, since $m_l = \sum_{\mathbf{S} \in \mathcal{S} : l \in \mathbf{S}} 1 \geq 2, \forall l \in \mathbf{S_0}$, we have $\sum_{\mathbf{H} \in \mathcal{H} : l \in \mathbf{HS_0}} 1 \geq 1, \forall l \in \mathbf{S_0}$. Then, we get

$$\sum_{l=1}^n P_l \leq n - \frac{\sum_{l=1}^n f(Q_l) \sum_{\mathbf{H} \in \mathcal{H} : l \in \mathbf{HS_0}} 1 + \sum_{\mathbf{T}:\mathbf{T} \in \mathcal{T}} |\mathbf{T} \setminus \mathbf{S_0}| f(0)}{2 \sum_{l=1}^n f(Q_l) \sum_{\mathbf{H} \in \mathcal{H} : l \in \mathbf{HS_0}} 1 + 2 \sum_{\mathbf{T}:\mathbf{T} \in \mathcal{T}} |\mathbf{T} \setminus \mathbf{S_0}| f(0)}$$
$$= n - \frac{1}{2}$$

Consider the Lyapunov function $L(\mathbf{Q}) := \sum_{i=1}^{n} Q_i$, then

$$\begin{aligned}&\Delta L(\mathbf{Q}) := \mathbb{E}\left[L(\mathbf{Q}[\mathbf{t+1}]) - L(\mathbf{Q}[\mathbf{t}])|\mathbf{Q}[\mathbf{t}] = \mathbf{Q}\right] \\ &= \mathbb{E}\left[\sum_{i=1}^{n}(Q_i[t] + A_i[t] - S_i[t] + U_i[t] - Q_i[t])|\mathbf{Q}[\mathbf{t}] = \mathbf{Q}\right] \\ &\geq \mathbb{E}\left[\sum_{i=1}^{n}(A_i[t] - S_i[t])|\mathbf{Q}[\mathbf{t}] = \mathbf{Q}\right] = \sum_{i=1}^{n}\lambda_i - \sum_{i=1}^{n} P_i\end{aligned}$$

For topologies in which each link belongs to two or more schedules, that is, $m_l = \sum_{\mathbf{S}\in\mathcal{S}:l\in\mathbf{S}} 1 \geq 2, \forall l \in \mathbf{S_0}$, if $\sum_{i=1}^{n}\lambda_i \geq n - \frac{1}{2}$, then $\Delta L(\mathbf{Q}) \geq 0$ for any $\mathbf{Q}$. Hence, by Theorem 20 of [11], the RSOF Scheduler is unstable if $\sum_{i=1}^{n}\lambda_i \geq n - \frac{1}{2}$ under such topologies.

(2) With the same model, the RFOS Scheduler becomes

$$\pi_{\mathbf{S}} = \frac{f(\sum_{l\in\mathbf{SS_0}} Q_l)}{\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{M}} f(\sum_{l\in\mathbf{SS_0}} Q_l) + \sum_{\mathbf{S}:\mathbf{S}\in\mathcal{K}} f(0)} \tag{18}$$

Then,

$$\sum_{l=1}^{n} P_l = \sum_{l=1}^{n}\sum_{\mathbf{S}\in\mathcal{M}:l\in\mathbf{SS_0}} \pi_{\mathbf{s}} = \frac{\sum_{l=1}^{n}\sum_{\mathbf{S}\in\mathcal{M}:l\in\mathbf{SS_0}} f(\sum_{i\in\mathbf{SS_0}} Q_i)}{\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{M}} f(\sum_{l\in\mathbf{SS_0}} Q_l) + \sum_{\mathbf{S}:\mathbf{S}\in\mathcal{K}} f(0)}$$

Since

$$\sum_{l=1}^{n}\sum_{\mathbf{S}\in\mathcal{M}:l\in\mathbf{SS_0}} f(\sum_{i\in\mathbf{SS_0}} Q_i) = \sum_{\mathbf{S}:\mathbf{S}\in\mathcal{M}} |\mathbf{SS_0}| f(\sum_{i\in\mathbf{SS_0}} Q_i)$$

we have

$$\begin{aligned}&\sum_{l=1}^{n} P_l = \frac{\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{M}} |\mathbf{SS_0}| f(\sum_{l\in\mathbf{SS_0}} Q_l)}{\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{M}} f(\sum_{l\in\mathbf{SS_0}} Q_l) + \sum_{\mathbf{S}:\mathbf{S}\in\mathcal{K}} f(0)} \\ &= \frac{n f(\sum_{l=1}^{n} Q_l) + \sum_{\mathbf{H}:\mathbf{H}\in\mathcal{H}} |\mathbf{HS_0}| f(\sum_{l\in\mathbf{HS_0}} Q_l)}{f(\sum_{l=1}^{n} Q_l) + \sum_{\mathbf{H}:\mathbf{H}\in\mathcal{H}} f(\sum_{l\in\mathbf{HS_0}} Q_l) + \sum_{\mathbf{S}:\mathbf{S}\in\mathcal{K}} f(0)} \\ &= n - \frac{\sum_{\mathbf{H}:\mathbf{H}\in\mathcal{H}}(n - |\mathbf{HS_0}|) f(\sum_{l\in\mathbf{HS_0}} Q_l) + n\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{K}} f(0)}{f(\sum_{l=1}^{n} Q_l) + \sum_{\mathbf{H}:\mathbf{H}\in\mathcal{H}} f(\sum_{l\in\mathbf{HS_0}} Q_l) + \sum_{\mathbf{S}:\mathbf{S}\in\mathcal{K}} f(0)}\end{aligned}$$

The fact that $f \in \mathcal{C}$ implies that there exist K_1 and K_2 satisfying $0 < K_1 \leq K_2 < \infty$ such that $K_1 \sum_{i=1}^{m} f(x_i) \leq f(\sum_{i=1}^{m} Q_i) \leq K_2 \sum_{i=1}^{m} f(Q_i)$, for $\forall m = 1, ..., n$, where $Q_i \geq 0, i = 1, ..., m$, which follows from induction. Then, we have

$$\begin{aligned}&\sum_{l=1}^{n} P_l \\ &\leq n - \frac{K_1}{K_2}\cdot\frac{\sum_{\mathbf{H}:\mathbf{H}\in\mathcal{H}}(n - |\mathbf{HS_0}|)\sum_{l\in\mathbf{HS_0}} f(Q_l) + n\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{K}} f(0)}{\sum_{l=1}^{n} f(Q_l) + \sum_{\mathbf{H}:\mathbf{H}\in\mathcal{H}}\sum_{l\in\mathbf{HS_0}} f(Q_l) + \sum_{\mathbf{S}:\mathbf{S}\in\mathcal{K}} f(0)} \\ &= n - \frac{K_1}{K_2}\cdot\frac{\sum_{l=1}^{n} f(Q_l)\sum_{\mathbf{H}\in\mathcal{H}:l\in\mathbf{HS_0}}(n - |\mathbf{HS_0}|) + n\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{K}} f(0)}{\sum_{l=1}^{n} f(Q_l) + \sum_{l=1}^{n} f(Q_l)\sum_{\mathbf{H}\in\mathcal{H}:l\in\mathbf{HS_0}} 1 + \sum_{\mathbf{S}:\mathbf{S}\in\mathcal{K}} f(0)}\end{aligned}$$

Note that $|\mathbf{HS_0}| \leq n - 1$, for $\forall \mathbf{H} \in \mathcal{H}$ and that $m_l = \sum_{\mathbf{S}\in\mathcal{S}:l\in\mathbf{S}} 1 \geq 2, \forall l \in \mathbf{S_0}$, implies that $\sum_{\mathbf{H}\in\mathcal{H}:l\in\mathbf{HS_0}} 1 \geq 1, \forall l \in \mathbf{S_0}$. Then, we get

$$\begin{aligned}\sum_{l=1}^{n} P_l &\leq n - \frac{K_1}{K_2}\cdot\frac{\sum_{l=1}^{n} f(Q_l)\sum_{\mathbf{S}\in\mathcal{M}':l\in\mathbf{SS_0}} 1 + \sum_{\mathbf{S}:\mathbf{S}\in\mathcal{N}} f(0)}{2\sum_{l=1}^{n} f(Q_l)\sum_{\mathbf{S}\in\mathcal{M}':l\in\mathbf{SS_0}} 1 + 2\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{N}} f(0)} \\ &\leq n - \frac{K_1}{2K_2}\end{aligned} \tag{19}$$

This shows that when $\min_{l\in\mathbf{S_0}} m_l \geq 2$ and $\sum_{i=1}^{n}\lambda_i \geq n - \frac{K_1}{2K_2}$, we have $\Delta L(\mathbf{Q}) \geq 0$ for any $\mathbf{Q}$. Hence, by Theorem 20 in paper [11], the RFOS Scheduler is unstable. ■

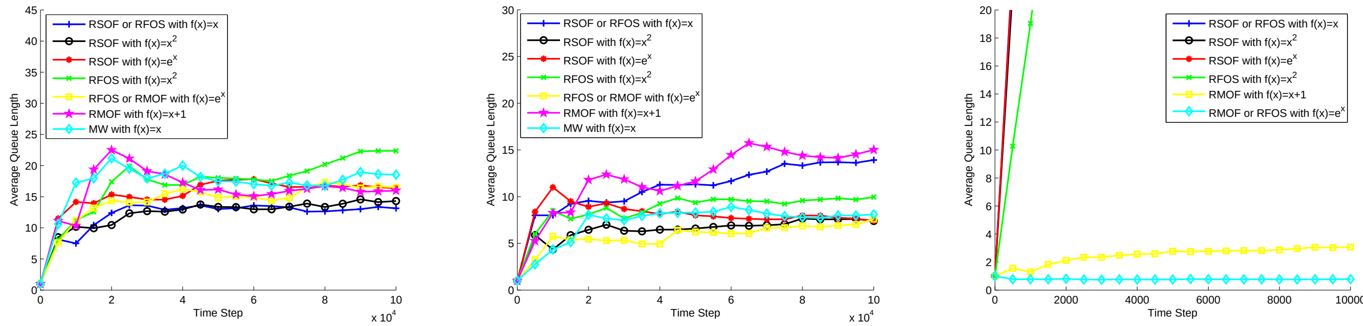
VI. Simulation Results

In this section, we perform numerical studies to validate the throughput performance of the proposed randomized schedulers with different functions in 2×2 and 3×3 switch topologies. In a 2×2 switch, the scheduling diversity of each link is 1 and thus all proposed randomized schedulers are proven to be throughput-optimal. In a 3×3 switch, the scheduling diversity of each link is 2, for which the RFOS Scheduler needs to carefully choose the functional form to preserve the throughput optimality while the RSOF Scheduler is not f-throughput-optimal with any function $f \in \mathcal{F}$

In a 2×2 switch, we consider arrival rate vector $\lambda = \rho\mathbf{H}$, where $\mathbf{H} = [H_{ij}]$ is a doubly-stochastic matrix with H_{ij} denoting the fraction of the total rate from input port i that is destined to output port j. Then, $\rho \in (0, 1)$ represents the average arrival intensity, where the larger the ρ, the more heavily loaded the switch is. We present two cases: symmetric arrival process ($\mathbf{H}_1 = [0.5\ 0.5; 0.5\ 0.5]$) and asymmetric arrival process ($\mathbf{H}_2 = [0.1\ 0.9; 0.9\ 0.1]$) under high arrival intensity $\rho = 0.99$.

From Figures 4(a) and 4(b), we can observe that all randomized schedulers can stabilize the system under symmetric and asymmetric arrival traffics. So, there is a wide class of choices under which the randomized scheduling can guarantee the throughput performance in the 2×2 switch. In addition, we can see that RSOF Scheduler with exponential function and RFOS Scheduler with square function are also stable in both symmetric and asymmetric arrival processes, which support our conjecture in Section III that RSOF Scheduler with the function $f \in \mathcal{A}$ and RFOS Scheduler with the function $f \in \mathcal{B}$ are f-throughput optimal in network topologies with $\max_l m_l \leq 1$.

In a 3×3 switch, we consider arrival rate vector $\lambda = [0.95\ 0\ 0; 0\ 0.95\ 0; 0\ 0\ 0.95]$, where RSOF Scheduler with any function $f \in \mathcal{F}$ and RFOS Scheduler with any function $f \in \mathcal{C}$ cannot stabilize. The evolution of average queue length per link over time for different schedulers with different functions are shown in figures 4(c). From Figure 4(c), we can observe that the average queue lengths of RSOF Schedulers with linear function, square function and even exponential function increase very fast, which validates our theoretical result that RSOF Scheduler with any function $f \in \mathcal{F}$ cannot be throughput-optimal in network topologies with $\max_l m_l \geq 2$. In addition, we can see that the average queue lengths of RFOS Schedulers with linear function and square function grow quickly while the RFOS Scheduler with exponential function always keeps low queue length level, which demonstrates that the steepness of functional form needs to be high enough for RFOS Scheduler to keep throughput optimality in general network topologies. Even though our result indicates that RMOF Scheduler with any function f satisfying $\log f \in \mathcal{B}$ and $f(0) \geq 1$ is $(\log f)$-throughput-optimal in general network topologies, we can see that RMOF Scheduler is still stable even with linear function. This validates that our conjecture that RMOF Scheduler with



(a) (b) (c)

Fig. 4: (a) Symmetric Arrivals in 2×2 Switch (b) Asymmetric Arrivals in 2×2 Switch (c) 3×3 Switch

any function $f \in \mathcal{F}$ can be f-throughput-optimal in general network topologies.

VII. Conclusions

We explored the limitations of randomization in the throughput-optimal scheduler design in a generic framework under the time-scale separation assumption. We identified three important functional forms of queue-length-based schedulers that covers a vast number of dynamic schedulers of interest. These forms differ fundamentally in whether they work with the queue-length of individual links or whole schedules.

For all of these functional forms, we established necessary and sufficient conditions on the network topology and the functional forms for their throughput-optimality. We also provided numerical results to validate our theoretical results and conjectures, which will be further studied in our future work.

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