Exploring the Throughput Boundaries of Randomized Schedulers in Wireless Networks

Bin Li and Atilla Eryilmaz

***Abstract*—Randomization is a powerful and pervasive strategy for developing efficient and practical transmission scheduling algorithms in interference-limited wireless networks. Yet, despite the presence of a variety of earlier works on the design and analysis of particular randomized schedulers, there does not exist an extensive study of the limitations of randomization on the efficient scheduling in wireless networks. In this work, we aim to fill this gap by proposing a common modeling framework and three functional forms of randomized schedulers that utilize queue-length information to probabilistically schedule non-conflicting transmissions. This framework not only models many existing schedulers operating under a time-scale separation assumption as special cases, but it also contains a much wider class of potential schedulers that have not been analyzed.**

We identify some sufficient and some necessary conditions on the network topology and on the functional forms used in the randomization for throughput-optimality. Our analysis reveals an *exponential* and a *sub-exponential* class of functions that exhibit differences in the throughput-optimality. Also, we observe the significance of the network's *scheduling diversity* for throughput-optimality as measured by the number of maximal schedules each link belongs to. We further validate our theoretical results through numerical studies.

***Index Terms*—Randomized scheduling, throughput-optimality, stochastic control, distributed algorithm, network stability.**

I. Introduction

One of the greatest challenges in the efficient communication in wireless networks is the management of interference amongst simultaneous transmissions. A commonly used model, which we also employ in this paper, to capture such interference effects is through the use of a *conflict graph* whereby transmissions that will collide with each other are indicated as conflicting. These conflict graphs can represent a variety of interference models of practical importance, including the primary interference model (e.g., [23], [9]), secondary interference model (e.g., [2], [3]), or SINR threshold-based interference model (e.g., [10]). Such conflict graphs can take on extremely complex forms, especially with growing network sizes. Thus, a fundamental question in the design of efficient wireless network protocols is the decision of which subset of non-conflicting transmissions to activate, and when - an operation commonly referred to as *scheduling*.

Of particular interest in the class of scheduling protocols is the set of *throughput-optimal* scheduling strategies (e.g., [26], [18]) that achieves any throughput (subject to network stability) that is achievable by any other scheduling strategy. Thus, throughput-optimal schedulers are critical especially for resource-limited wireless networks as they achieve the largest possible throughput region that is supportable by the network. The seminal works of Tassiulas and Ephremides [26], [27] and many subsequent works (e.g., [4], [18], [24]; see [5] for an overview) have established the throughput-optimality of a variety of *Queue-Length-Based (QLB) Scheduling* strategies, which prioritize activation of links with the greatest backlog awaiting service, also called *Maximum Weight Scheduling (MWS)*.

These original throughput-optimal strategies require the maximum weight schedule to be determined repeatedly as the queue-length levels change. This calls for computationally heavy (even NP-hard in certain interference models) and typically centralized operations, which is impractical. Such restrictions have motivated new research efforts to develop more practical throughput-optimal schedulers with reduced complexity. One such thread led to the development of a class of evolutionary randomized algorithms (also named *pick and compare* algorithms) with throughput-optimality characteristics (see [25], [3], [22]). Another thread led to the development of distributed but suboptimal randomized/greedy strategies (see [13], [8], [1]).

More recently, another exciting thread of results have emerged that can guarantee throughput-optimality by cleverly utilizing queue-length information in the context of carrier sense multiple access (CSMA) (see [14], [7], [20], [19]). In paper [7], the authors proposed an algorithm that adaptively selects the CSMA parameters under a time-scale separation assumption, i.e., the Markov Chain underlying the CSMA-based algorithm converges to steady-state quickly compared with the time-scale of updating parameters of the algorithm. In paper [21], the authors showed the throughput-optimality of a CSMA-based algorithm in which the link weights are chosen to be of the form $\log\log(q+e)$ (where q is the queue length) without the time-scale separation assumption. Ghaderia and Srikant [6] extended these results by showing that the throughput-optimality of CSMA-based algorithm will be preserved even if the link weights have the form $\log(q)/g(q)$, where $g(q)$ can be a function that increases to infinity arbitrarily slowly. Yet, to the best of our knowledge, there does not exist a general framework in which a variety of randomized schedulers can be

This work was supported in part by Qatar National Research Fund (QNRF) under the National Research Priorities Program (NPRP) grant number NPRP 09-1168-2-455, DTRA Grant HDTRA 1-08-1-0016, and NSF Awards: CAREER-CNS-0953515 and CCF-0916664.

An earlier version of this paper has appeared in the Proceedings of 30th IEEE International Conference on Computer Communications (INFOCOM), Shanghai, China, April 2011 [12].

Bin Li and Atilla Eryilmaz ({lib,eryilmaz}@ece.osu.edu) are with the Department of Electrical and Computer Engineering at The Ohio State University, Columbus, Ohio 43210 USA.

studied in terms of their throughput-optimality characteristics.

Thus, in this work, we aim to fill this gap by developing a common framework for the modeling and analysis of queue-length-based randomized schedulers, and then by establishing necessary and sufficient conditions on the throughput-optimality of a large functional class of such schedulers under the time-scale separation assumption. Our framework is built upon the observation that a common characteristic to most of the developed schedulers is their randomized selection of transmission schedules from the set of all feasible schedules. Specifically, given the existing queue-lengths of the links, each scheduling strategy can be viewed as a particular probability distribution over the set of feasible schedules. While the means with which this random assignment may vary in its distributiveness or complexity, this perspective allows us to model a large set of existing and an even wider set of potential randomized schedulers within a common framework.

This work builds on this original point-of-view to explore the boundaries of randomization in the throughput-optimal operation of wireless networks. Such an investigation is crucial in revealing the necessary and sufficient characteristics of randomized schedulers and the network topologies in which throughput-optimality can be achieved.

Next, we list our main contributions along with references on where they appear in the text.

• In Section II, we highlight the pressing need for developing new randomized schedulers, for example, for operation under fading conditions and for serving delay-related application requirements. We also note with a specific example that these new schedulers may possess fundamentally different probabilistic operation than existing distributed solutions with product form mappings. This motivates us to study the performance limitations of wide class of randomization strategies.

• In Section II, we introduce three functional forms of randomized queue-length-based scheduling strategies that include many existing strategies as special cases (see Definitions 3, 4 and 5). These strategies differ in the manner in which they measure the weight of schedules, and hence are used to model fundamentally different scheduling implementations.

• We categorize the set of all functions used by these strategies into functions of *exponential form* and of *sub-exponential form* (see Definition 6), collectively covering almost all functions of interest. These two categories capture the steepness of the functions used in the schedulers, and help reveal a critical degree of steepness necessary for throughput-optimality in large networks.

• Then, we find some sufficient (in Section IV) and some necessary (in Section V) conditions on the topological characteristics of the conflict graph for the throughput-optimality of these schedulers as a function of the class of functions used in their operation. Our results, graphically summarized in Section III, reveal the significance of the network's scheduling diversity that is measured by the number of schedules each link belongs to.

II. System Model

A. Basic Definitions

We consider a fixed wireless network represented by a graph $\mathcal{G} = (\mathcal{N}, \mathcal{L})$, where $\mathcal{N}$ is the set of nodes and $\mathcal{L}$ is the set of undirected links. We assume a time-slotted system, where all nodes transmit at the beginning of each time slot. Due to the interference-limited nature of wireless transmissions, the success or failure of a transmission over a link depends on whether an *interfering* link is also active in the same slot. For ease of exposition, we assume that a successful transmission over any link in each slot transfers one packet.

We use *conflict graphs* to capture any such collision-based interference in the wireless networks. In a *conflict graph* $\mathcal{CG} = (\mathcal{L}, \mathcal{E})$ of $\mathcal{G}$ under a given interference model, the set of links $\mathcal{L}$ in $\mathcal{G}$ becomes the set of nodes, and $\mathcal{E}$ denotes the set of edges that connects links that interfere with each other. In each time slot, we can successfully transmit over nodes in a subset of $\mathcal{L}$ that form an *independent set* (i.e., that are not directly connected in $\mathcal{CG}$). We call each such independent set as a *feasible schedule*, and denote it as $\mathbf{S} = (S_l)_{l \in \mathcal{L}}$, where $S_l = 1$ if link l is active and $S_l = 0$ is link l is inactive in the schedule. We also treat $\mathbf{S}$ as a set of active links and write $l \in \mathbf{S}$ if $S_l = 1$. We use $|\mathbf{S}|$ to denote the cardinality of the set $\mathbf{S}$. We further call a feasible schedule as *maximal* if no more nodes in $\mathcal{CG}$ can be added without violating the interference constraint. As maximal schedules represent extreme points in the space of feasible schedules, we collect them in the set $\mathcal{S}$. Then, we can define the *capacity region* Λ as the convex hull[1] of $\mathcal{S}$ and L-dimensional all-zero vector, which will give the upper bound on the achievable link rates in packets per slot that can be supported by the network under stability for the given interference model.

Given the topology and the interference model of a wireless network, we define the *scheduling diversity of link* $l \in \mathcal{L}$ as the number of different maximal schedules m_l that link l belongs to. Since each link $l \in \mathcal{L}$ belongs to at least one maximal schedule, m_l should be the integer greater than or equal to 1. For a network topology with a complete N-partite conflict graph[2], we have $m_l = 1, \forall l \in \mathcal{L}$. As another example, a single-hop wireless network where all links interfere with each other, we have $m_l = 1$ for all l. Less trivially, a 2×2 switch has 2-partite conflict graph in which each maximal schedule has only 2 links, and $m_l = 1$ for each l. Roughly speaking, the scheduling diversity increases as the *network diameter*[3] increases. Such a behavior can be observed directly in a linear network with L links under the primary interference model: for $L \leq 3$, $m_l = 1$ for all l; for $L \geq 6$, $m_l \geq 2$ for all l.

In its simplest form, a *scheduler* determines a maximal feasible schedule $\mathbf{S}[t] \in \mathcal{S}$ at each time slot t. This selection

[1] The convex hull of the set $\mathbf{V}$ is the minimal convex set containing set $\mathbf{V}$.

[2] In a complete N-partite conflict graph, the nodes are partitioned into N sets of nodes without a link between them such that every node in each set is connected to all the nodes outside of that set.

[3] Network diameter is the maximum of the shortest hop-count between any two nodes in the graph.

may be influenced by the earlier experiences of each transmitter, and may be performed through a variety of strategies. Here, we are not interested in the means of selecting schedules, but in the eventual selection modeled as a probabilistic function of the queue-length state of the network. Before we define the class of randomized schedulers we consider more explicitly, we need to establish the traffic and the queueing models.

For simplicity, we assume a per-link traffic model[4], where $A_l[t]$ arrivals occur to link l in slot t that are independently distributed over links and identically distributed over time with mean λ_l, and $A_l[t] \leq K$ for some $K < \infty$ [5]. Accordingly, a queue is maintained for each link $l \in \mathcal{L}$ with $Q_l[t]$ denoting its queue length at the beginning of time slot t. Recall from above that $S_l[t]$ denotes the number of potential departures at time t. Further, we let $U_l[t]$ denote the unused service for the queue l in slot t. If the queue l is empty and is scheduled, then $U_l[t]$ is equal to 1; otherwise, it is equal to 0. Then, the evolution of the queue l is described as follows:

$$Q_l[t+1] = Q_l[t] + A_l[t] - S_l[t] + U_l[t], \qquad \forall l \in \mathcal{L}. \quad (1)$$

We define $\mathcal{F}$:= set of non-negative, nondecreasing and differentiable functions $f(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{x\to\infty} f(x) = \infty$. We say that the queue l is *f-stable* for a function $f \in \mathcal{F}$ if it satisfies

$$\limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[f(Q_l[t])] < \infty. \quad (2)$$

We note that this is an extended form of the more traditional strong stability condition (see [5]) that coincide when $f(x) = x$. Moreover, it is easy to show that f-stability implies strong stability when f is also a convex function. We say that the *network is f-stable* if all its queues are f-stable. Accordingly, we say that a scheduler is *f-throughput-optimal* if it achieves f-stability of the network for any arrival rate vector $\lambda = (\lambda_l)_{l\in\mathcal{L}}$ that lies strictly inside the capacity region Λ. Again, in the special case of $f(x) = x$, the notion of f-throughput-optimality reduces to traditional throughput-optimality, and when f is convex, f-throughput-optimality implies throughput-optimality.

B. Distributed Algorithms

The operation of many existing schedulers are governed by probabilistic laws (e.g., [25], [13], [14], [7], [20], [3]). This is not only because they model possible errors in the scheduling process, but also because they allow significant flexibilities in the development of low-complexity and distributed implementations. Of particular interest in this class of probabilistic schedulers are distributed CSMA-based algorithms (e.g., [7], [15], [19]). We give the definition of continuous time CSMA algorithm for completeness.

[4]This assumption can be relaxed by utilizing backpressure type routing strategy (see, for example, [26]), which is avoided for unnecessary complications.

[5]We note that the boundedness assumption on the arrival process simplifies the technical arguments, but can be relaxed (see, for example, [4]) to the less strict assumption of $E[A_l^2(t)] < \infty$.

Definition 1 (CSMA Algorithm): Each link l independently generates an exponentially distributed random variable with rate $f(Q_l[t])$ and starts transmitting after this random duration unless it senses another transmission before. If link l senses the transmission, it suspends its backoff timer and resumes it after the completion of this transmission. The transmission time of each link is exponential distributed with mean 1. $\diamond$

A common characteristics of these CSMA-based schedulers is the product form (see Definition 4) of the mapping of the total queue-length levels to the probability of the associated schedule. Such a mapping has been observed to closely approximate the operation of the throughput-optimal centralized MWS [26], and hence also possesses throughput optimality characteristics. However, these CSMA-based algorithm cannot be directly extended to operate under stochastic network dynamics or sophisticated application requirements. As an important example, extending CSMA solutions to serving traffic with strict deadline constraints under wireless fading channels is difficult for two reasons: (1) the mixing time of the underlying CSMA Markov Chain grows with the size of the network, which, for large networks, generates unacceptable delay for deadline-constrained traffic; (2) since the dynamic CSMA parameters are influenced by the arrival and channel state process, the underlying CSMA Markov Chain may not converge to a steady-state under strict deadline constrains and fading channel conditions.

Thus, designing an optimal distributed scheduling algorithm in deadline-constrained scheduling over fading channels becomes very challenging. In a recent work [11], we have found that, in some special network topologies, the following Fast carrier sensing multiple access (FCSMA) algorithm can guarantee optimal performance.

Definition 2 (FCSMA Algorithm): At the beginning of each time slot t, each link l independently generates an exponentially distributed random variable with rate $f(Q_l[t])$, and starts transmitting after this random duration unless it senses another transmission before. The link that captures the channel transmits its packets[6] until the end of the slot and restarts in the next time slot. $\diamond$

We note that FCSMA differs from CSMA in that it restarts every time slot, and hence increases the probability of meeting deadline requirements. Further remarks on FCSMA are shown as follows:

Remarks: (1) Consider a complete N-partite conflict graph, where each link only belongs to one schedule. If all queue lengths are large enough, then the idle duration in each slot will quickly vanish, and FCSMA algorithm reaches one of the maximal schedule and stick to it for one time slot. Thus, the FCSMA algorithm serving a schedule $\mathbf{S}$ with probability

$$P_{\mathbf{S}} = \frac{\sum_{i\in\mathbf{S}} f(Q_i)}{\sum_{\{\mathbf{S}':\mathbf{S}'\in\mathcal{S}\}} \sum_{j\in\mathbf{S}'} f(Q_j)} \quad (3)$$

[6]If there are no packets awaiting in the link l, it transmits a dummy packet to occupy the channel.

is f-throughput-optimal, which is proven in Section IV. An interesting observation is that this scheduler does not approximate the MWS operation when queue-lengths are large, as CSMA does. For example, consider a 2×2 switch topology, where there are only two maximal schedules $\mathbf{S}_1$ including two active links l_1 and l_2 and $\mathbf{S}_2$ including two active links l_3 and l_4. Suppose all queue lengths are large enough and $Q_{l_1} + Q_{l_2} = 0.5(Q_{l_3} + Q_{l_4})$, then the MWS chooses the schedule $\mathbf{S}_2$ and CSMA algorithm selects the schedule $\mathbf{S}_2$ with probability very close to 1. However, FCSMA policy chooses the schedule $\mathbf{S}_2$ with probability close to $2/3$. This indicates the importance of understanding schedulers with fundamentally different behavior than MWS.

(2) In a fully connected network topology, due to its fast absorption time and quick adaptation to arrival and channel state processes, FCSMA policy yields significant advantages over traditional CSMA policies that evolves slowly to their steady-state, especially in scheduling deadline constrained traffic over wireless fading channels. We refer the interested reader to [11] for more detailed investigation of FCSMA operation.

It is also worth noting that for a given stationary distribution, it is possible to construct a Markov Chain that converges to it by Metropolis algorithm [16] or Glauber dynamics (e.g., [21], [6]). Yet, in this paper, we do not focus on the design of specific scheduling algorithms that can converge to the stationary distribution. Instead, we are interested in the throughput-optimality characteristics of a wide class of probabilistic mapping from the queue length space to the feasible schedules.

In the following, we consider three classes of randomized schedulers which not only model many existing probabilistic schedulers as special cases but also contain a much wider classes of potential schedulers that have not been analyzed.

C. Randomized Schedulers

In this subsection, we identify three classes of randomized schedulers that differ in the operation of the functional forms used in them.

Definition 3 (RSOF Scheduler): For a given $f \in \mathcal{F}$ and queue-length vector $\mathbf{Q}$ at the beginning of a slot, the Ratio-of-Sum-of-Functions (RSOF) Scheduler picks a schedule $\mathbf{S} \in \mathcal{S}$ in that slot such that

$$P_{\mathbf{S}}(\mathbf{Q}) := \frac{\sum_{i \in \mathbf{S}} f(Q_i)}{\sum_{\{\mathbf{S}':\mathbf{S}' \in \mathcal{S}\}} \sum_{j \in \mathbf{S}'} f(Q_j)}, \text{ for all } \mathbf{S} \in \mathcal{S}. \quad (4)$$

Definition 4 (RMOF Scheduler): For a given $f \in \mathcal{F}$ and queue-length vector $\mathbf{Q}$ at the beginning of a slot, the Ratio-of-Multiplication-of-Functions (RMOF) Scheduler picks a schedule $\mathbf{S} \in \mathcal{S}$ in that slot such that

$$\upsilon_{\mathbf{S}}(\mathbf{Q}) := \frac{\prod_{i \in \mathbf{S}} f(Q_i)}{\sum_{\{\mathbf{S}':\mathbf{S}' \in \mathcal{S}\}} \prod_{j \in \mathbf{S}'} f(Q_j)}, \text{ for all } \mathbf{S} \in \mathcal{S}. \quad (5)$$

Definition 5 (RFOS Scheduler): For a given $f \in \mathcal{F}$ and queue-length vector $\mathbf{Q}$ at the beginning of a slot, the Ratio-of-Function-of-Sums (RFOS) Scheduler picks a schedule $\mathbf{S} \in \mathcal{S}$ in that slot such that

$$\pi_{\mathbf{S}}(\mathbf{Q}) := \frac{f(\sum_{i \in \mathbf{S}} Q_i)}{\sum_{\{\mathbf{S}':\mathbf{S}' \in \mathcal{S}\}} f(\sum_{j \in \mathbf{S}'} Q_j)}, \text{ for all } \mathbf{S} \in \mathcal{S}. \quad (6)$$

Note that all the RSOF, RMOF and RFOS Schedulers are more likely to pick a schedule with the larger queue length, but with different distributions based on their form and the form of $f \in \mathcal{F}$. In particular, the steepness of the function f determines the weight given to the heavily loaded *link* in both RSOF and RMOF Schedulers and the heavily loaded *schedule* in the RFOS Scheduler. Also, note that the schedulers coincide in single-hop network topologies because each maximal schedule only includes one link in such networks, and for the following choices of f: when $f(x) = x$, the RSOF and RFOS Schedulers coincide; when $f(x) = e^x$, the RMOF and RFOS Schedulers coincide. These three classes cover a wide variety of schedulers including many of existing throughput-optimal schedulers. For example, when $f(x) = e^x$, the RMOS and RFOS Schedulers correspond to the throughput-optimal CSMA policy operating under time-scale separation assumption that attracted a lot of attention lately (see [7], [20], [19]); in a complete N-partite conflict graph, the RSOF Scheduler corresponds to the FCSMA policy when all the queue lengths are large enough. Yet, they also contain a much wider set of schedulers, one for each f.

The aim of this work is to identify the limitations of randomization for a wide class of randomized dynamic schedulers that utilize functions of queue-lengths to schedule transmissions. Even though randomization has significant advantage in low-complexity or distributed implementation, it causes inaccurate operation and may be hurtful if not performed within limitations. In this work, we find that the performance of the randomized schedulers may especially be sensitive to the topology of the conflict graph and the functional form used in the weighting. To see this, consider one maximal schedule $\mathbf{S}_1$ including three active links l_1, l_2 and l_3 in a 3×3 switch topology. We assume that arrivals only happen to those 3 links at rates λ_{l_1}, λ_{l_2} and λ_{l_3} with the constraints that $\lambda_{l_i} \in [0, 1)$ for all $i = 1, 2, 3$, which clearly can be supported by a simple policy that always serves the schedule $\mathbf{S}_1$. Thus, by setting λ_{l_i} arbitrarily close to one for each i, this simple policy can achieve a sum rate of $\sum_{i=1}^{3} \lambda_{l_i} < 3$. However, for a RFOS Scheduler with $f(x) = x$, we can easily calculate that $\sum_{i=1}^{3} \theta_{l_i} = 2$, where θ_{l_i} $(i = 1, 2, 3)$ is the probability of serving link l_i. Thus, the RFOS Scheduler with $f(x) = x$ cannot achieve full capacity region in a 3×3 switch.

Yet, in the same set up, if we use $f(x) = e^x$ instead of $f(x) = x$ in the RFOS Scheduler, the mapping has the same probabilistic form as the CSMA policy, and thus would be throughput-optimal. This shows the significant impact of the functional form on the throughput performance of randomized schedulers. In addition, the RFOS Scheduler with $f(x) = x$ is shown to be f-throughput-optimal in a 2×2 switch (see Figure 3), which indicates that the network topology may also affect the throughput performance of randomized schedulers.

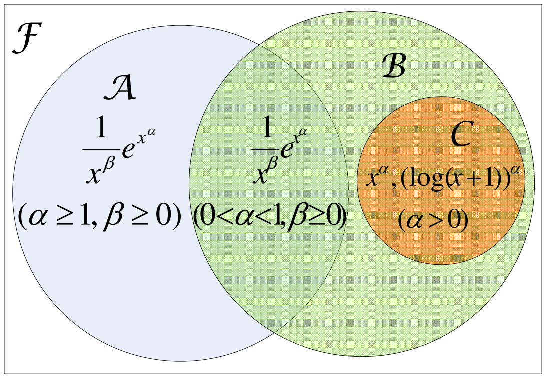


Fig. 1: The relationship between classes $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.

Next, we identify the three classes of functions with varying forms that turn out to be crucial to our investigation.

Definition 6: We consider the following subsets of $\mathcal{F}$:

(a) $\mathcal{A} := \{f \in \mathcal{F} : \forall \epsilon > 0, \lim_{x\to\infty} \frac{f(x)}{f((1+\epsilon)x)} = 0\}$.

(b) $\mathcal{B} := \{f \in \mathcal{F}$: $\lim_{x\to\infty} \frac{f(x+a)}{f(x)} = 1$, for any $a \in \mathbb{R}\}$.

(c) $\mathcal{C} := \{f \in \mathcal{B}$: there exist K_1 and K_2 satisfying $0 < K_1 \leq K_2 < \infty$ such that $K_1(f(x_1) + f(x_2)) \leq f(x_1 + x_2) \leq K_2(f(x_1) + f(x_2))$, for all $x_1, x_2 \geq 0\}$.

We call $\mathcal{A}$ as the *class of exponential functions* and $\mathcal{C}$ as the *class of sub-exponential functions*. The key examples of functions with sets $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and their interrelationship are extensively studied in Appendix A.

Figure 1 concisely demonstrates the most critical facts: that $\mathcal{A}$ and $\mathcal{C}$ are non-overlapping classes; while $\mathcal{B}$ has an intersection with $\mathcal{A}$. Furthermore, the example functions are provided with a variety of forms that justify the names assigned to $\mathcal{A}$ and $\mathcal{C}$: $\mathcal{A}$ contains rapidly increasing functions generally with exponential forms; while $\mathcal{C}$ contains sub-exponentially increasing polynomial and logarithmic functional forms. In the study of necessary and sufficient conditions for throughput-optimality, we shall find that most of the results depend on which of these three functional classes the functions belong to.

III. Overview of Main Results

In this section, we present our main findings and resulting insights on the throughput-optimality of the RSOF, RMOF and RFOS Schedulers (see Definitions 3, 4 and 5) with different functional forms under different network topologies. These results are rigorously proven in Sections IV and V. To facilitate an accessible figurative presentation, in the horizontal dimension, we conceptually order the functions in $\mathcal{F}$ in increasing level of steepness starting from $f(x) = (\log(x+1))^\alpha$ and $f(x) = x^\alpha$ for any $\alpha > 0$ that belong to $\mathcal{C}$, followed by $f(x) = \frac{1}{x^\beta}e^{x^\alpha}$ for any $0 < \alpha < 1$ and any $\beta \geq 0$ that belongs to $\mathcal{B} \bigcap \mathcal{A}$, and finishing with $f(x) = \frac{1}{x^\beta}e^{x^\alpha}$ for any $\alpha \geq 1$ and any $\beta \geq 0$ that belongs to $\mathcal{A}$. In the vertical dimension, we use the scheduling diversity $(m_l)_{l\in\mathcal{L}}$ introduced in Section II to distinguish different topological and interference scenarios. Recall that since m_l denotes the number of different maximal schedules that link l belongs to, it may be viewed as a *rough* measure of the network diameter. Then, the main results for the RSOF and RFOS Schedulers are presented in Figures 2 and 3, respectively. In these figures, we also include several conjectures that are validated through simulations in Section VI.

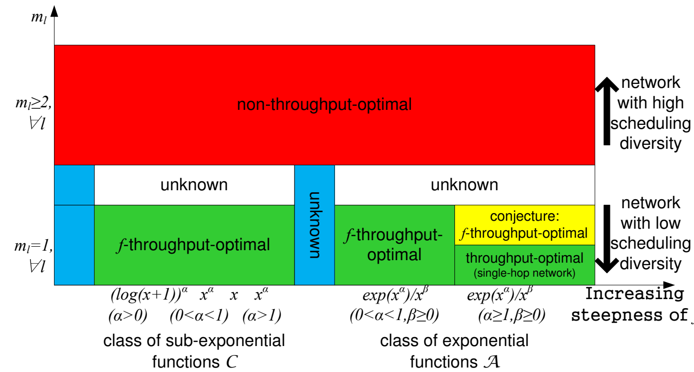


Fig. 2: The throughput performance of the RSOF Scheduler.

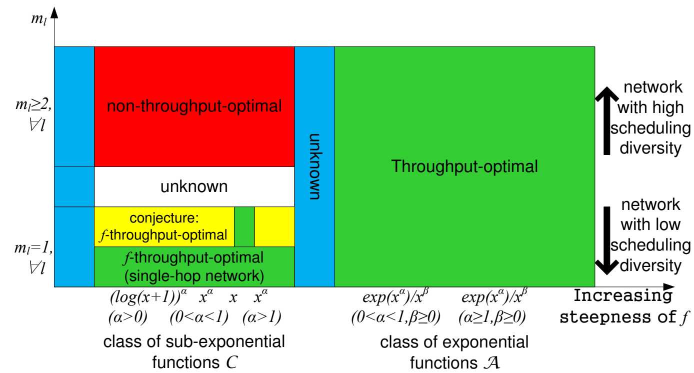


Fig. 3: The throughput performance of the RFOS Scheduler.

From Figure 2, we see that the RSOF Scheduler with the function $f \in \mathcal{B}$ is f-throughput-optimal when $m_l = 1, \forall l \in \mathcal{L}$. Also, the RSOF Scheduler with the function $f \in \mathcal{A} \setminus \mathcal{B}$ is throughput-optimal in single-hop network topologies since the RSOF and RFOS Schedulers have the same probability distribution over schedules in such networks and the RFOS Scheduler with the function $f \in \mathcal{A}$ is throughput-optimal (see Figure 3). However, if $\min_{l\in\mathcal{L}} m_l \geq 2$, the RSOF Scheduler with any function $f \in \mathcal{F}$ cannot be throughput-optimal. Thus, roughly speaking, the RSOF Scheduler is non-throughput-optimal for the network with high scheduling diversity, while the RSOF Scheduler with the function $f \in \mathcal{B}$ is f-throughput-optimal for low scheduling diversity. We note that although the throughput performance of the RSOF Scheduler with some exponential functions $f \in \mathcal{A} \setminus \mathcal{B}$ (i.e. $f(x) = \frac{1}{x^\beta}e^{x^\alpha}$, $\alpha \geq 1$ and $\beta \geq 0$) is not yet explored in general topologies with $m_l = 1, \forall l \in \mathcal{L}$, we conjecture that it is f-throughput-optimal in this region, since the RSOF Scheduler with such functions reacts much more quickly to the queue length difference between schedules than that with sub-exponential functions, especially under asymmetric arrival patterns. We validate this conjecture through simulations in Section VI. Overall, the RSOF Scheduler is more sensitive to the network topology than the functional form used in it.

The horizontal unknown region corresponds to network topologies where some links have scheduling diversity 1 and other links have scheduling diversity at least 2. The vertical unknown region corresponds to randomized schedulers with functions that are not in the functional classes $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. In Figure 3, we observe that the RFOS Scheduler with the function $f \in \mathcal{A}$ is throughput-optimal under any network topology. Also, the RFOS Scheduler with the function $f \in \mathcal{C}$ is f-throughput-optimal in single-hop network topologies, which follows from the fact that the RFOS and RSOF Schedulers have the same probability probabilistic forms in such networks, the result that the RSOF Scheduler with the function $f \in \mathcal{B}$ is f-throughput-optimal (see Figure 2) and the fact that $\mathcal{C} \subseteq \mathcal{B}$. Also, when the function f is linear, the RFOS Scheduler has the same probability form with the RSOF Scheduler and thus is f-throughput-optimal when $m_l = 1, \forall l \in \mathcal{L}$. However, the RFOS Scheduler with the function $f \in \mathcal{C}$ is not throughput-optimal when $\min_{l \in \mathcal{L}} m_l \geq 2$. Roughly speaking, the network with higher scheduling diversity requires much steeper functions (e.g., exponential functions) for the throughput-optimality of the RFOS Scheduler. While the throughput performance of the RFOS Scheduler with the function $f \in \mathcal{C} \setminus \{\text{linear functions}\}$ for general network topologies with $m_l = 1, \forall l \in \mathcal{L}$ is part of our ongoing work, we conjecture that it is f-throughput-optimal in those topologies since both RFOS and RSOF Schedulers with sub-exponential functions have almost the same reaction speed to the queue length difference between schedules. We also validate this conjecture via simulations in Section VI. Overall, the RFOS Scheduler is more sensitive to the functional form used in it than the network topology.

The RMOF Scheduler with the function f satisfying $\log f \in \mathcal{B}$ and $f(0) \geq 1$ is $(\log f)$-throughput-optimal under any network topology. This result together with the RFOS Scheduler with the function $f \in \mathcal{A}$ extends the throughput-optimality of CSMA schedulers (e.g. [7], [19]) to a wider class of functional forms. While this result proves a weaker form of throughput-optimality than f-throughput-optimality for the RMOF Scheduler, we note that the RMOF Scheduler generally outperforms the RFOS and RSOF Schedulers in numerical investigations. Hence, we leave it to future research to strengthen this result.

Collectively these results not only highlight the strengths and weaknesses of the three functional randomized schedulers, they also reveal the interrelation between the steepness of the functions and the scheduling diversity of the underlying wireless networks. This extensive understanding of the limitations of randomization may motivate the network designers to use or avoid certain types of probabilistic scheduling strategies depending on the topological characteristics of the network.

IV. Sufficient Conditions

In this section, we study the sufficient conditions on the network's topological characteristics and the functions used in the RSOF, RMOF and RFOS Schedulers to achieve throughput-optimality.

A. f-Throughput-Optimality of the RSOF Scheduler

We study the throughput performance of the RSOF Scheduler for a network topology with $m_l = 1, \forall l \in \mathcal{L}$. In such a network, each link belongs to only one maximal schedule.

Lemma 1: If $\sum_{i=1}^{N} \lambda_i < 1$, $\lambda_i > 0$, and $a_i \geq 0$, for $i = 1, ..., N$, then there exists a $\delta > 0$ such that

$$\sum_{i=1}^{N} \frac{a_i^2}{\lambda_i} \geq (\sum_{i=1}^{N} a_i)^2 (1 + \delta). \tag{7}$$

Proof: See Appendix B for the proof. ■

Theorem 1: In a network topology with the scheduling diversity of each link equal to 1, i.e., $m_l = 1, \forall l \in \mathcal{L}$, the RSOF Scheduler with the function $f \in \mathcal{B}$ is f-throughput-optimal.

Proof: We assume that there are only N available maximal schedules. Let $\mathbf{S}^i$ $(i = 1, ..., N)$ denote the i^{th} maximal schedule. In each maximal schedule $\mathbf{S}^i$, there are $|\mathbf{S}^i|$ active links. We use $(\mathbf{S}_l^i,\ l = 1, ..., |\mathbf{S}^i|)$ to denote the sequence of active links in the maximal schedule $\mathbf{S}^i$. Note that we use i to index maximal schedule and l to index link. Since the schedule diversity of each link is equal to 1, each link belongs to only one maximal schedule. Thus, we can denote the queues, arrivals, and scheduling statistics in terms of maximal schedules for easier exposition. To that end, we let Q_l^i, λ_l^i and P_l^i $(i = 1, ..., N, l = 1, ..., |\mathbf{S}^i|)$ denote the queue-length of link $l \in \mathbf{S}^i$, the average arrival rate for the link $l \in \mathbf{S}^i$ and the probability of serving the link $l \in \mathbf{S}^i$, respectively. In addition, $A_l^i[t]$, $S_l^i[t]$ and $U_l^i[t]$ denote the number of arrivals to link $l \in \mathbf{S}^i$ at time slot t, the number of potential departures of link $l \in \mathbf{S}^i$ in slot t and the unused service for link $l \in \mathbf{S}^i$ at time slot t, respectively. Recall that each link can only belong to one maximal schedule and note that links in different maximal schedules cannot be active at the same time. Thus, the capacity region for such network is

$$C_N := \{\lambda : \sum_{i=1}^{N} \lambda_{l_i}^i < 1, \forall l_i = 1, ..., |\mathbf{S}^i|\}. \tag{8}$$

Under the above notation, the RSOF Scheduler becomes :

$$P_{\mathbf{S}^i} = \frac{\sum_{l=1}^{|\mathbf{S}^i|} f(Q_l^i)}{\sum_{k=1}^{N} \sum_{l=1}^{|\mathbf{S}^k|} f(Q_l^k)}, i = 1, ..., N. \tag{9}$$

Note that $P_l^i = P_{\mathbf{S}^i}$, for $l = 1, ..., |\mathbf{S}^i|$. If $\lambda_l^i = 0$ for some i and l, then no arrivals occur in the link $l \in \mathbf{S}^i$. Thus, we don't need to consider such links. In the rest of proof, we assume $\lambda_l^i > 0$ $(i = 1, ..., N,\ l = 1, ..., |\mathbf{S}^i|)$. Consider the Lyapunov function $V(\mathbf{Q}) := \sum_{i=1}^{N} \sum_{l=1}^{|\mathbf{S}^i|} \frac{h(Q_l^i)}{\lambda_l^i}$, where $h'(x) = f(x)$. By using Lemma 1, it is shown in the Appendix C that there exist positive constants γ and G such that

$$\Delta V := \mathbb{E}\left[V(\mathbf{Q}[\mathbf{t+1}]) - V(\mathbf{Q}[\mathbf{t}]) | \mathbf{Q}[t] = \mathbf{Q}\right] \leq -\gamma \sum_{i=1}^{N} \sum_{l=1}^{|\mathbf{S}^i|} f(Q_l^i) + G. \quad (10)$$

By using the Theorem 4.1 in [17], inequality (10) implies the desired result. ■

B. Throughput-Optimality of RMOF and RFOS Schedulers

In this subsection, we investigate the sufficient condition for the throughput-optimality of RMOF and RFOS Schedulers.

Theorem 2: (i) The RMOF Scheduler with the function $f \in \mathcal{F}$ satisfying $\log f \in \mathcal{B}$ and $f(0) \geq 1$ is $(\log f)$-throughput-optimal under any network topology;
(ii) The RFOS Scheduler with the function $f \in \mathcal{A}$ is throughput-optimal under any network topology.

Proof: To prove this, we use a similar approach as in [19] that uses the following result from [4]: for a scheduling algorithm, given any $0 \leq \epsilon, \delta < 1$, there exists an $M > 0$ for which the scheduling algorithm satisfies the following condition: in any time slot t, with probability greater than $1-\delta$, the scheduling algorithm chooses a schedule $\mathbf{x}[t] \in \mathcal{S}$ that satisfies: $\sum_{l \in \mathbf{x}[t]} w(Q_l[t]) \geq (1-\epsilon) \max_{\mathbf{x} \in \mathcal{S}} \sum_{l \in \mathbf{x}[t]} w(Q_l[t])$, whenever $\parallel \mathbf{Q}[\mathbf{t}] \parallel > M$, where $\mathbf{Q}[t] := (Q_l[t])_{l \in \mathcal{L}}$, and $w \in \mathcal{B}$. Then the scheduling algorithm is w-throughput-optimal.
(i) Given any ϵ_1 and δ_1 such that $0 \leq \epsilon_1, \delta_1 < 1$. Let

$$\mathcal{X}_1 := \{\mathbf{x} \in \mathcal{S} : \sum_{l \in \mathbf{x}} \log f(Q_l[t]) < (1-\epsilon_1) W_1^*[t]\}, \quad (11)$$

where $W_1^*[t] := \max_{\mathbf{x} \in \mathcal{S}} \sum_{l \in \mathbf{x}} \log f(Q_l[t])$. Then, we have

$$\begin{aligned} \upsilon(\mathcal{X}_1) &= \sum_{\mathbf{x} \in \mathcal{X}_1} \upsilon_{\mathbf{x}} = \sum_{\mathbf{x} \in \mathcal{X}_1} \frac{\prod_{l \in \mathbf{x}} f(Q_l[t])}{\sum_{\mathbf{x}' \in \mathcal{S}} \prod_{l \in \mathbf{x}'} f(Q_l[t])} \\ &= \frac{\sum_{\mathbf{x} \in \mathcal{X}_1} \exp\left[\sum_{l \in \mathbf{x}} \log f(Q_l[t])\right]}{\sum_{\mathbf{x} \in \mathcal{S}} \exp\left[\sum_{l \in \mathbf{x}} \log f(Q_l[t])\right]} \\ &< \frac{|\mathcal{X}_1| \exp\left[(1-\epsilon_1) W_1^*[t]\right]}{\sum_{\mathbf{x} \in \mathcal{S}} \exp\left[\sum_{l \in \mathbf{x}} \log f(Q_l[t])\right]}. \end{aligned}$$

Since $\sum_{\mathbf{x} \in \mathcal{S}} \exp\left[\sum_{l \in \mathbf{x}} \log f(Q_l[t])\right] \geq \exp(W_1^*[t])$, then we get

$$\upsilon(\mathcal{X}_1) < \frac{|\mathcal{X}_1| \exp\left[(1-\epsilon_1) W_1^*[t]\right]}{\exp(W_1^*[t])} = \frac{|\mathcal{X}_1|}{\exp(\epsilon_1 W_1^*[t])}. \quad (12)$$

If some queue lengths increase to infinity, then $W_1^*[t] \to \infty$ and thus we have $\upsilon(\mathcal{X}_1) \to 0$. Hence, there exists a $M_1 > 0$ such that $\parallel \mathbf{Q}[t] \parallel > M_1$ and the RMOF Scheduler with the function $f \in \mathcal{F}$ satisfying $\log f \in \mathcal{B}$ and $f(0) \geq 1$ picks the schedule $\mathbf{S}[t] \in \mathcal{S} \setminus \mathcal{X}_1$ with probability $1-\delta_1$ and thus is $\log f$-throughput-optimal under any topology.
(ii) Given any ϵ_2 and δ_2 such that $0 \leq \epsilon_2, \delta_2 < 1$. Let $W_2^*[t] := \max_{\mathbf{x} \in \mathcal{S}} \sum_{l \in \mathbf{x}} Q_l[t]$, and $\mathcal{X}_2 := \{\mathbf{x} \in \mathcal{S} : \sum_{l \in \mathbf{x}} Q_l[t] < (1-\epsilon_2) W_2^*[t]\}$. Then, by using the same technique as in (i), we can prove that the RFOS Scheduler with $f \in \mathcal{A}$ is throughput-optimal under any topology. ■

V. Necessary Conditions

So far, we have shown that the RSOF Scheduler with the function $f \in \mathcal{B}$ is f-throughput-optimal in the network topology with $m_l = 1, \forall l \in \mathcal{L}$ and the RFOS Scheduler with the function $f \in \mathcal{A}$ is throughput-optimal under arbitrary network topologies. However, the next result establishes that in network topologies where each link belongs to two or more schedules (i.e. when $\min_{l \in \mathcal{L}} m_l \geq 2$), the RSOF Scheduler with any function $f \in \mathcal{F}$ and RFOS Scheduler with the function $f \in \mathcal{C}$ cannot be throughput-optimal.

Theorem 3: If the network is such that $\min_{l \in \mathcal{L}} m_l \geq 2$, then
(i) RSOF Scheduler is not throughput-optimal for any $f \in \mathcal{F}$;
(ii) RFOS Scheduler is not throughput-optimal for any $f \in \mathcal{C}$.

Proof: We prove these claims constructively by considering an arrival process that is inside the capacity region, but is not supportable by the randomized schedulers for the given functional forms. To that end, let us consider any maximal schedule $\mathbf{S}_0 \in \mathcal{S}$ and index its links as $\{1, 2, ..., n\}$ for convenience. We assume that arrivals only happen to those n links at rates $\lambda_1, \cdots, \lambda_n$ with the constraint that $\lambda_l \in [0, 1)$ for all $l = 1, \cdots, n$, which is clearly supportable by a simple scheduling policy that always serves the schedule $\mathbf{S}_0$. Thus, setting λ_l arbitrarily close to one for each l, this simple policy can achieve a sum rate of $\sum_{l=1}^{n} \lambda_l < n$.

We define $\mathcal{M} = \{\mathbf{S} \in \mathcal{S} : \mathbf{S} \bigcap \mathbf{S_0} \neq \emptyset\}$, $\mathcal{K} = \mathcal{S} \setminus \mathcal{M}$, $\mathcal{H} = \mathcal{M} \setminus \{\mathbf{S_0}\}$ and $\mathcal{T} = \mathcal{S} \setminus \{\mathbf{S_0}\}$. In the rest of the proof, we use $\mathbf{AB}$ to denote the intersection of $\mathbf{A}$ and $\mathbf{B}$.

Given this construction, we next prove the following statements for the RSOF and RFOS Schedulers respectively:
(1) If $\sum_{l=1}^{n} \lambda_l > n - \frac{1}{2}$, the RSOF Scheduler with any function $f \in \mathcal{F}$ is unstable.
(2) If $\sum_{l=1}^{n} \lambda_l > n - \frac{K_1'}{2K_2'}$, where K_1' and K_2' are positive constants described in Appendix A, the RFOS Scheduler with the associated function $f \in \mathcal{C}$ is unstable.

Since the aforementioned simple scheduler can stabilize the sum rate $\sum_{l=1}^{n} \lambda_l < n$, the RSOF Scheduler with any function $f \in \mathcal{F}$ and RFOS Scheduler with the associated function $f \in \mathcal{C}$ are not throughput-optimal. We next prove these claims that complete the proof of Theorem 3.
(1) Under the above model, the RSOF Scheduler becomes

$$P_{\mathbf{S}} = \frac{\sum_{l \in \mathbf{S}\mathbf{S_0}} f(Q_l) + |\mathbf{S} \setminus \mathbf{S_0}| f(0)}{\sum_{\mathbf{S}' : \mathbf{S}' \in \mathcal{S}} (\sum_{l \in \mathbf{S}'\mathbf{S_0}} f(Q_l) + |\mathbf{S}' \setminus \mathbf{S_0}| f(0))}.$$

Let P_l denote the probability that link $l \in \mathbf{S}_0$ is served, then

$$\sum_{l=1}^{n} P_l = \sum_{l=1}^{n} \sum_{\mathbf{S} \in \mathcal{M} : l \in \mathbf{S}\mathbf{S_0}} P_{\mathbf{S}} = \frac{\overbrace{\sum_{l=1}^{n} \sum_{\mathbf{S} \in \mathcal{M} : l \in \mathbf{S}\mathbf{S_0}} \left(\sum_{i \in \mathbf{S}\mathbf{S_0}} f(Q_i) + |\mathbf{S} \setminus \mathbf{S_0}| f(0) \right)}^{=:L_1}}{\underbrace{\sum_{\mathbf{S} : \mathbf{S} \in \mathcal{S}} \sum_{l \in \mathbf{S}\mathbf{S_0}} f(Q_l) + \sum_{\mathbf{S} : \mathbf{S} \in \mathcal{S}} |\mathbf{S} \setminus \mathbf{S_0}| f(0)}_{=:L_2}}.$$

Since $\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{S}}\sum_{l\in\mathbf{SS_0}} f(Q_l)=\sum_{l=1}^{n} f(Q_l)\sum_{\mathbf{S}\in\mathcal{M}:l\in\mathbf{SS_0}}1$, $\sum_{l=1}^{n}\sum_{\mathbf{S}\in\mathcal{M}:l\in\mathbf{SS_0}}|\mathbf{S}\setminus\mathbf{S_0}|f(0)=\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{S}}|\mathbf{SS_0}||\mathbf{S}\setminus\mathbf{S_0}|f(0)$, and

$$\sum_{l=1}^{n}\sum_{\mathbf{S}\in\mathcal{M}:l\in\mathbf{SS_0}}\sum_{i\in\mathbf{SS_0}} f(Q_i)=\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{M}}\sum_{l\in\mathbf{SS_0}}\sum_{i\in\mathbf{SS_0}} f(Q_i)$$
$$=\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{M}}|\mathbf{SS_0}|\sum_{i\in\mathbf{SS_0}} f(Q_i)=\sum_{l=1}^{n} f(Q_l)\sum_{\mathbf{S}\in\mathcal{M}:l\in\mathbf{SS_0}}|\mathbf{SS_0}|,$$

we can extend L_1 and L_2 as follows:

$$L_1=\sum_{l=1}^{n} f(Q_l)\sum_{\mathbf{S}\in\mathcal{M}:l\in\mathbf{SS_0}}|\mathbf{SS_0}|+\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{S}}|\mathbf{SS_0}||\mathbf{S}\setminus\mathbf{S_0}|f(0)$$
$$=\sum_{l=1}^{n} f(Q_l)(n+\sum_{\mathbf{H}\in\mathcal{H}:l\in\mathbf{HS_0}}|\mathbf{HS_0}|)+\sum_{\mathbf{T}:\mathbf{T}\in\mathcal{T}}|\mathbf{TS_0}||\mathbf{T}\setminus\mathbf{S_0}|f(0),$$

and

$$L_2=\sum_{l=1}^{n} f(Q_l)\sum_{\mathbf{S}\in\mathcal{M}:l\in\mathbf{SS_0}}1+\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{S}}|\mathbf{S}\setminus\mathbf{S_0}|f(0)$$
$$=\sum_{l=1}^{n} f(Q_l)(1+\sum_{\mathbf{H}\in\mathcal{H}:l\in\mathbf{HS_0}}1)+\sum_{\mathbf{T}:\mathbf{T}\in\mathcal{T}}|\mathbf{T}\setminus\mathbf{S_0}|f(0).$$

Thus, we have

$$\sum_{l=1}^{n} P_l=\frac{L_1}{L_2}=n-\frac{Z_1}{Z_2}, \qquad (13)$$

where $Z_1=\sum_{l=1}^{n} f(Q_l)\sum_{\mathbf{H}\in\mathcal{H}:l\in\mathbf{HS_0}}(n-|\mathbf{HS_0}|)+\sum_{\mathbf{T}:\mathbf{T}\in\mathcal{T}}(n-|\mathbf{TS_0}|)|\mathbf{T}\setminus\mathbf{S_0}|f(0)$, and $Z_2=\sum_{l=1}^{n} f(Q_l)(1+\sum_{\mathbf{H}\in\mathcal{H}:l\in\mathbf{HS_0}}1)+\sum_{\mathbf{T}:\mathbf{T}\in\mathcal{T}}|\mathbf{T}\setminus\mathbf{S_0}|f(0)$. Note that $|\mathbf{HS_0}|\le n-1$, for $\forall\mathbf{H}\in\mathcal{H}$, and $|\mathbf{TS_0}|\le n-1$, for $\forall\mathbf{T}\in\mathcal{T}$. Now, since $m_l=\sum_{\mathbf{S}\in\mathcal{S}:l\in\mathbf{S}}1\ge 2,\forall l\in\mathbf{S_0}$, we have $\sum_{\mathbf{H}\in\mathcal{H}:l\in\mathbf{HS_0}}1\ge 1,\forall l\in\mathbf{S_0}$. Then, we get

$$\frac{Z_1}{Z_2}\ge\frac{\sum_{l=1}^{n} f(Q_l)\sum_{\mathbf{H}\in\mathcal{H}:l\in\mathbf{HS_0}}1+\sum_{\mathbf{T}:\mathbf{T}\in\mathcal{T}}|\mathbf{T}\setminus\mathbf{S_0}|f(0)}{2\sum_{l=1}^{n} f(Q_l)\sum_{\mathbf{H}\in\mathcal{H}:l\in\mathbf{HS_0}}1+2\sum_{\mathbf{T}:\mathbf{T}\in\mathcal{T}}|\mathbf{T}\setminus\mathbf{S_0}|f(0)}$$
$$=\frac{1}{2}.$$

Thus, we have $\sum_{l=1}^{n} P_l\le n-\frac{1}{2}$. Hence, for topologies where $\min_{l\in\mathcal{L}} m_l\ge 2$, if $\sum_{l=1}^{n}\lambda_l>n-\frac{1}{2}$, in which case the total arrival rate is greater than the total service rate, then, the RSOF Scheduler is unstable by following the Theorem 2.8 and Theorem 2.5 in [17].

(2) With the same model, the RFOS Scheduler becomes

$$\pi_{\mathbf{S}}=\frac{f(\sum_{l\in\mathbf{SS_0}}Q_l)}{\sum_{\mathbf{S'}:\mathbf{S'}\in\mathcal{M}} f(\sum_{l\in\mathbf{S'S_0}}Q_l)+\sum_{\mathbf{S''}:\mathbf{S''}\in\mathcal{K}} f(0)}. \qquad (14)$$

Then,

$$\sum_{l=1}^{n} P_l=\sum_{l=1}^{n}\sum_{\mathbf{S}\in\mathcal{M}:l\in\mathbf{SS_0}}\pi_{\mathbf{s}}$$
$$=\frac{\sum_{l=1}^{n}\sum_{\mathbf{S}\in\mathcal{M}:l\in\mathbf{SS_0}} f(\sum_{i\in\mathbf{SS_0}}Q_i)}{\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{M}} f(\sum_{l\in\mathbf{SS_0}}Q_l)+\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{K}} f(0)}.$$

Since

$$\sum_{l=1}^{n}\sum_{\mathbf{S}\in\mathcal{M}:l\in\mathbf{SS_0}} f(\sum_{i\in\mathbf{SS_0}}Q_i)=\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{M}}|\mathbf{SS_0}|f(\sum_{i\in\mathbf{SS_0}}Q_i),$$

we have

$$\sum_{l=1}^{n} P_l=\frac{\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{M}}|\mathbf{SS_0}|f(\sum_{l\in\mathbf{SS_0}}Q_l)}{\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{M}} f(\sum_{l\in\mathbf{SS_0}}Q_l)+\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{K}} f(0)}$$
$$=\frac{nf(\sum_{l=1}^{n}Q_l)+\sum_{\mathbf{H}:\mathbf{H}\in\mathcal{H}}|\mathbf{HS_0}|f(\sum_{l\in\mathbf{HS_0}}Q_l)}{f(\sum_{l=1}^{n}Q_l)+\sum_{\mathbf{H}:\mathbf{H}\in\mathcal{H}} f(\sum_{l\in\mathbf{HS_0}}Q_l)+\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{K}} f(0)}$$
$$=n-\frac{\sum_{\mathbf{H}:\mathbf{H}\in\mathcal{H}}(n-|\mathbf{HS_0}|)f(\sum_{l\in\mathbf{HS_0}}Q_l)+n\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{K}} f(0)}{f(\sum_{l=1}^{n}Q_l)+\sum_{\mathbf{H}:\mathbf{H}\in\mathcal{H}} f(\sum_{l\in\mathbf{HS_0}}Q_l)+\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{K}} f(0)}.$$

The fact that $f\in\mathcal{C}$ implies that there exist K'_1 and K'_2 satisfying $0<K'_1\le K'_2<\infty$ such that $K'_1\sum_{i=1}^{m} f(Q_i)\le f(\sum_{i=1}^{m}Q_i)\le K'_2\sum_{i=1}^{m} f(Q_i)$, for $\forall m=1,...,n$, where $Q_i\ge 0, i=1,...,m$, which follows from induction. Then, we have

$$\sum_{l=1}^{n} P_l$$
$$\le n-\frac{K'_1}{K'_2}\cdot\frac{\sum_{\mathbf{H}:\mathbf{H}\in\mathcal{H}}(n-|\mathbf{HS_0}|)\sum_{l\in\mathbf{HS_0}} f(Q_l)+n\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{K}} f(0)}{\sum_{l=1}^{n} f(Q_l)+\sum_{\mathbf{H}:\mathbf{H}\in\mathcal{H}}\sum_{l\in\mathbf{HS_0}} f(Q_l)+\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{K}} f(0)}$$
$$=n-\frac{K'_1}{K'_2}\cdot\frac{\sum_{l=1}^{n} f(Q_l)\sum_{\mathbf{H}\in\mathcal{H}:l\in\mathbf{HS_0}}(n-|\mathbf{HS_0}|)+n\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{K}} f(0)}{\sum_{l=1}^{n} f(Q_l)+\sum_{l=1}^{n} f(Q_l)\sum_{\mathbf{H}\in\mathcal{H}:l\in\mathbf{HS_0}}1+\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{K}} f(0)}.$$

Note that $|\mathbf{HS_0}|\le n-1$, for $\forall\mathbf{H}\in\mathcal{H}$, and that $m_l=\sum_{\mathbf{S}\in\mathcal{S}:l\in\mathbf{S}}1\ge 2,\forall l\in\mathbf{S_0}$, implies that $\sum_{\mathbf{H}\in\mathcal{H}:l\in\mathbf{HS_0}}1\ge 1,\forall l\in\mathbf{S_0}$. Then, we get

$$\sum_{l=1}^{n} P_l\le n-\frac{K'_1}{K'_2}\cdot\frac{\sum_{l=1}^{n} f(Q_l)\sum_{\mathbf{H}\in\mathcal{H}:l\in\mathbf{HS_0}}1+\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{K}} f(0)}{2\sum_{l=1}^{n} f(Q_l)\sum_{\mathbf{H}\in\mathcal{H}:l\in\mathbf{HS_0}}1+2\sum_{\mathbf{S}:\mathbf{S}\in\mathcal{K}} f(0)}$$
$$\le n-\frac{K'_1}{2K'_2}. \qquad (15)$$

Thus, by following the same argument as in the proof for statement (1), we know that when $\min_{l\in\mathcal{L}} m_l\ge 2$ and $\sum_{l=1}^{n}\lambda_l>n-\frac{K'_1}{2K'_2}$, the RFOS Scheduler is unstable. ■

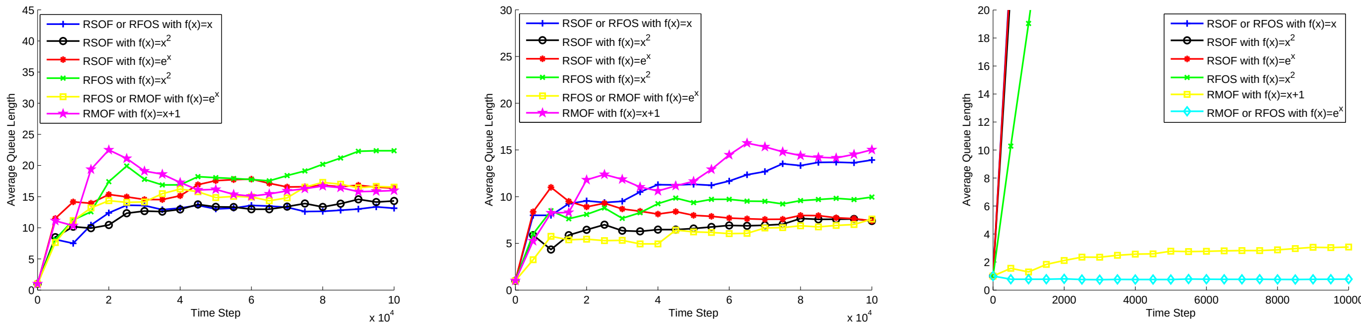
VI. Simulation Results

In this section, we first perform numerical studies to validate the throughput performance of the proposed randomized schedulers with different functions in 2×2 and 3×3 switch topologies. Then, we evaluate the impact of functional forms on the delay performance of proposed randomized schedulers in 2×2 switch topologies.

A. Throughput Performance

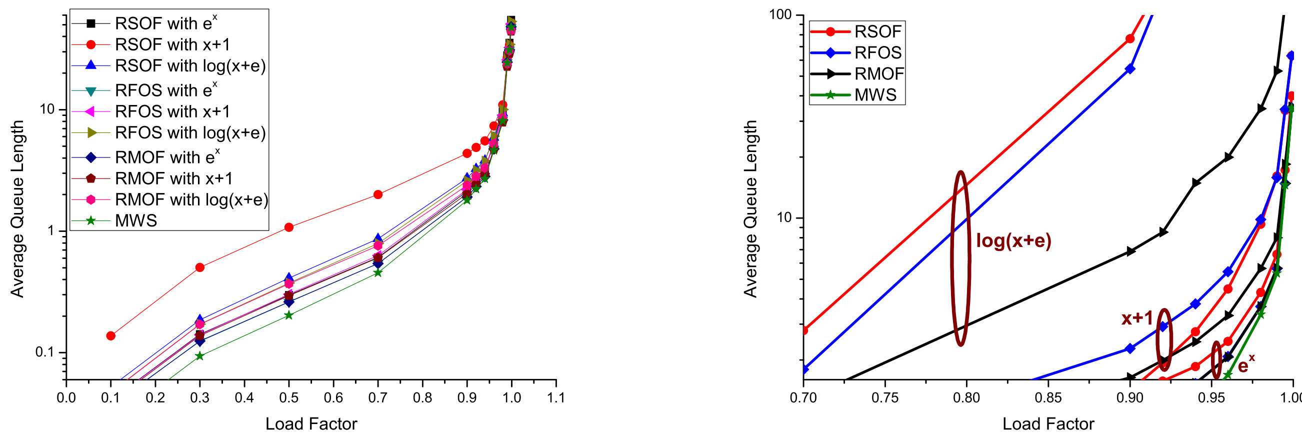
In a 2×2 switch, the scheduling diversity of each link is 1 and thus all proposed randomized schedulers are proven to be throughput-optimal. In a 3×3 switch, the scheduling diversity of each link is 2, for which the RFOS Scheduler needs to carefully choose the functional form to preserve the throughput optimality while the RSOF Scheduler is not f-throughput-optimal with any function $f\in\mathcal{F}$

In a 2×2 switch, we consider arrival rate vector $\lambda=\rho\mathbf{H}$, where $\mathbf{H}=[H_{ij}]$ is a doubly-stochastic matrix with H_{ij}



(a) Symmetric arrivals in a 2×2 switch (b) Asymmetric arrivals in a 2×2 switch (c) 3×3 switch

Fig. 4: The throughput performance validation of the randomized schedulers



(a) Symmetric arrivals in a 2×2 switch (b) Asymmetric arrivals in a 2×2 switch

Fig. 5: Delay performance comparison of the randomized schedulers with different functional forms

denoting the fraction of the total rate from input port i that is destined to output port j. Then, $\rho \in (0,1)$ represents the average arrival intensity, where the larger the ρ, the more heavily loaded the switch is. We present two cases: symmetric arrival process ($\mathbf{H}_1 = [0.5\ 0.5; 0.5\ 0.5]$) and asymmetric arrival process ($\mathbf{H}_2 = [0.1\ 0.9; 0.9\ 0.1]$) under high arrival intensity $\rho = 0.99$.

From Figures 4(a) and 4(b), we can observe that all randomized schedulers can stabilize the system under symmetric and asymmetric arrival traffics. So, there is a wide class of choices under which the randomized scheduling can guarantee the throughput performance in the 2×2 switch. In addition, we can see that the RSOF Scheduler with the exponential function and the RFOS Scheduler with the square function are also stable in both symmetric and asymmetric arrival processes, which support our conjecture in Section III that the RSOF Scheduler with the function $f \in \mathcal{A}$ and the RFOS Scheduler with the function $f \in \mathcal{B}$ are f-throughput optimal in network topologies with $m_l = 1, \forall l \in \mathcal{L}$.

In a 3×3 switch, we consider arrival rate vector $\lambda = [0.95\ \ 0\ \ 0; 0\ \ 0.95\ \ 0; 0\ \ 0\ \ 0.95]$, where the RSOF Scheduler with any function $f \in \mathcal{F}$ and the RFOS Scheduler with any function $f \in \mathcal{C}$ cannot stabilize. The evolution of average queue length per link over time for different schedulers with different functions are shown in figures 4(c). From Figure 4(c), we can observe that the average queue lengths of the RSOF Schedulers with linear function, square function and even exponential function increase very fast, which validates our theoretical result that the RSOF Scheduler with any function $f \in \mathcal{F}$ cannot be throughput-optimal in network topologies with $\min_{l \in \mathcal{L}} m_l \geq 2$. In addition, we can see that the average queue lengths of the RFOS Schedulers with linear function and square function grow quickly while the RFOS Scheduler with exponential function always keeps low queue length level, which demonstrates that the steepness of functional form needs to be high enough for the RFOS Scheduler to keep throughput optimality in general network topologies. Even though our result indicates that the RMOF Scheduler with any function

f satisfying $\log f \in \mathcal{B}$ and $f(0) \geq 1$ is $(\log f)$-throughput-optimal in general network topologies, we can see that the RMOF Scheduler is still stable even with linear function. This validates that our conjecture that the RMOF Scheduler with any function $f \in \mathcal{F}$ can be f-throughput-optimal in general network topologies.

B. Delay Performance

In this subsection, we perform numerical studies to evaluate the delay performance of proposed randomized schedulers with different functions in a 2×2 switch topology.

From Figure 5(a), we can observe that, under symmetric arrival traffic, the delay performance is highly insensitive to the choice of the randomization and the functional form being used in it especially under high arrival load. So, there is a wide class of choices under which the randomized scheduling can yield good delay performance. On the other hand, Figure 5(b) demonstrates that, under asymmetric arrival traffic, the RMOF Scheduler is more robust to the choice of functions used in it than both the RSOF and RFOS Schedulers. In particular, it appears that the steepness of f needs to be high enough for each randomization to yield good delay performance. Generally, the RMOF Scheduler outperforms the other two randomized schedulers especially under asymmetric arrival traffic. In all cases, the RSOF and RFOS Schedulers have similar performance and MWS has the best delay performance.

While these numerical studies indicate a number of interesting facts on the mean delay performance of randomized schedulers, we leave a more careful delay performance comparison to future research. There is clearly a need for a deeper investigation of delay performance of throughput-optimal schedulers. This work forms the foundation to investigate these higher-order performance metrics in our future research.

VII. Conclusions

We explored the limitations of randomization in the throughput-optimal scheduler design in a generic framework under the time-scale separation assumption. We identified three important functional forms of queue-length-based schedulers that covers a vast number of dynamic schedulers of interest. These forms differ fundamentally in whether they work with the queue-length of individual links or whole schedules.

For all of these functional forms, we established some sufficient and some necessary conditions on the network topology and the functional forms for their throughput-optimality. We also provided numerical results to validate our theoretical results and conjectures, which will be further studied in our future work.

Appendix A
Properties of functional classes

The following remarks explore more properties of classes $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.

(1) In $\mathcal{B}$, if $\lim_{x\to\infty} \frac{f(x+a)}{f(x)}$ exists for any $a \in \mathbb{R}$, then this limit should be equal to 1. Indeed, let $\lim_{x\to\infty} \frac{f(x+a)}{f(x)} = b$ for any $a \in \mathbb{R}$, where $b > 0$. Then $b = \lim_{x\to\infty} \frac{f(x+2)}{f(x)} = \lim_{x\to\infty} \frac{f(x+2)}{f(x+1)} \cdot \frac{f(x+1)}{f(x)} = b^2$. Thus, $b = 1$.

(2) If the definition of $\mathcal{C}$ is not constrained by the set $\mathcal{B}$, then $\mathcal{C}$ is not necessarily a subset of $\mathcal{B}$. In fact, we can construct a function $f \in \mathcal{C}$ for which $\lim_{x\to\infty} \frac{f(x+a)}{f(x)}$ does not exist and hence $f \notin \mathcal{B}$.

(3) In $\mathcal{C}$, if $f \in \mathcal{F}$, then the lower bound of $f(x_1 + x_2)$ always exists. Also if there exists $w > 0$ such that $f(2x) \leq wf(x)$ for any $x \geq 0$, then the upper bound of $f(x_1 + x_2)$ always exists. Indeed, since $f(\cdot)$ is nondecreasing, $f(x_1 + x_2) \geq f(x_i)$, for $i = 1$ or 2. Hence $f(x_1 + x_2) \geq \frac{1}{2}(f(x_1) + f(x_2))$. Thus, let $K_1 = \frac{1}{2}$, then we always have $K_1(f(x_1)+f(x_2)) \leq f(x_1+x_2)$. On the other hand, $f(x_1 + x_2) \leq \max\{f(2x_1), f(2x_2)\} \leq f(2x_1) + f(2x_2) \leq w(f(x_1) + f(x_2))$. Thus, let $K_2 = w$, we have $f(x_1 + x_2) \leq K_2(f(x_1) + f(x_2))$.

(4) If $f \in \mathcal{C}$, then given $n \in \mathbb{N}$, there exist K_1' and K_2' satisfying $0 < K_1' \leq K_2' < \infty$ such that $K_1' \sum_{i=1}^m f(x_i) \leq f(\sum_{i=1}^m x_i) \leq K_2' \sum_{i=1}^m f(x_i)$, for $m = 1, ..., n$, where $x_i \geq 0, i = 1, ..., m$. This directly follows from the induction.

(5) $\mathcal{A} \bigcap \mathcal{C} = \emptyset$. Indeed, if $f \in \mathcal{A}$, then $\lim_{x\to\infty} \frac{f(2x)}{f(x)} = \infty$. Thus, for any $c > 0$, $\exists M > 0$ such that $f(2x) > cf(x)$ for any $x > M$. Hence, $f \notin \mathcal{C}$. On the other hand, if $f \in \mathcal{C}$, then $\exists d > 0$ such that $f(2x) \leq df(x)$. Hence, $\limsup_{x\to\infty} \frac{f(2x)}{f(x)} \leq d$ and thus $f \notin \mathcal{A}$.

Appendix B
Proof for Lemma 1

Proof: If $n = 1$, because $\lambda_1 \in (0, 1)$, by assumption, there exists a $0 < \delta_1 < \frac{1}{\lambda_1} - 1$, such that $\frac{a_1^2}{\lambda_1} \geq a_1^2(1 + \delta_1)$.

Assume that $n = k$, it is true. That is, if $\sum_{i=1}^k \lambda_i < 1$ and $\lambda_i > 0$ $(i = 1, ..., k)$, then there exists a $\delta_k = \delta(\lambda_1, ..., \lambda_k) > 0$ such that

$$\frac{1}{\lambda_1}a_1^2 + ... + \frac{1}{\lambda_k}a_k^2 \geq (a_1 + ... + a_k)^2(1 + \delta_k). \tag{16}$$

Then for $n = k + 1$ and $\lambda_1 + ... + \lambda_k + \lambda_{k+1} < 1$, we have

$$\begin{aligned}
&\frac{1}{\lambda_1}a_1^2 + ... + \frac{1}{\lambda_k}a_k^2 + \frac{1}{\lambda_{k+1}}a_{k+1}^2 \\
=&\frac{1}{\lambda_1}a_1^2 + ... + \frac{1}{\lambda_{k-1}}a_{k-1}^2 \\
&+ \frac{1}{\lambda_k + \lambda_{k+1}}(\frac{\lambda_k + \lambda_{k+1}}{\lambda_k}a_k^2 + \frac{\lambda_k + \lambda_{k+1}}{\lambda_{k+1}}a_{k+1}^2) \\
\geq &\left[a_1 + ... + a_{k-1} + \sqrt{\frac{\lambda_k + \lambda_{k+1}}{\lambda_k}a_k^2 + \frac{\lambda_k + \lambda_{k+1}}{\lambda_{k+1}}a_{k+1}^2}\right]^2 \\
&\cdot (1 + \delta_{k+1}) \text{ (by assumption).}
\end{aligned} \tag{17}$$

Since

$$
\begin{aligned}
&\frac{\lambda_k+\lambda_{k+1}}{\lambda_k}a_k^2+\frac{\lambda_k+\lambda_{k+1}}{\lambda_{k+1}}a_{k+1}^2-(a_k+a_{k+1})^2\\
=&\frac{\lambda_{k+1}}{\lambda_k}a_k^2+\frac{\lambda_k}{\lambda_{k+1}}a_{k+1}^2-2a_ka_{k+1}\\
\geq&2\sqrt{\frac{\lambda_{k+1}}{\lambda_k}a_k^2\cdot\frac{\lambda_k}{\lambda_{k+1}}a_{k+1}^2}-2a_ka_{k+1}=0, \qquad (18)
\end{aligned}
$$

hence

$$
\sqrt{\frac{\lambda_k+\lambda_{k+1}}{\lambda_k}a_k^2+\frac{\lambda_k+\lambda_{k+1}}{\lambda_{k+1}}a_{k+1}^2}\geq(a_k+a_{k+1}). \qquad (19)
$$

Thus, equation (17) becomes

$$
\sum_{i=1}^{k+1}\frac{1}{\lambda_i}a_i^2\geq(\sum_{i=1}^{k+1}a_i)^2(1+\delta_{k+1}).
$$

■

APPENDIX C
PROOF OF INEQUALITY (10)

$$
\begin{aligned}
\Delta V:=&\,\mathbb{E}\left[V(\mathbf{Q}[\mathbf{t+1}])-V(\mathbf{Q}[\mathbf{t}])|\mathbf{Q}[t]=\mathbf{Q}\right]\\
=&\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\mathbb{E}\left[\frac{1}{\lambda_l^i}(h(Q_l^i[t+1])-h(Q_l^i[t]))|\mathbf{Q}[t]=\mathbf{Q}\right].
\end{aligned}
$$

By the mean-value theorem, we have $h(Q_l^i[t+1])-h(Q_l^i[t])=f(R_l^i[t])(Q_l^i[t+1]-Q_l^i[t])=f(R_l^i[t])(A_l^i[t]-S_l^i[t]+U_l^i[t])$, where $R_l^i[t]$ lies between $Q_l^i[t]$ and $Q_l^i[t+1]$. Hence, we get

$$
\begin{aligned}
\Delta V=&\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\mathbb{E}\left[\frac{1}{\lambda_l^i}f(R_l^i[t])(A_l^l[t]-S_l^i[t]+U_l^i[t])|\mathbf{Q}[t]=\mathbf{Q}\right]\\
=&\underbrace{\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\mathbb{E}\left[\frac{1}{\lambda_l^i}f(R_l^i[t])U_l^i[t]|\mathbf{Q}[t]=\mathbf{Q}\right]}_{=:\Delta V_1}+\\
&\underbrace{\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\mathbb{E}\left[\frac{1}{\lambda_l^i}f(R_l^i[t])(A_l^i[t]-S_l^i[t])|\mathbf{Q}[t]=\mathbf{Q}\right]}_{=:\Delta V_2}.
\end{aligned}
$$

For ΔV_1, if $Q_l^i[t]=Q_l^i>0$, then $U_l^i[t]=0$. If $Q_l^i[t]=Q_l^i=0$, then $U_l^i[t]$ may be equal to 1. But in this case, $Q_l^i[t+1]\leq K$ (since $A_l^i[t]\leq K$). Hence, $f(R_l^i[t])\leq f(K)<\infty$. Thus,

$$
\begin{aligned}
\Delta V_1=&\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\mathbb{E}\left[\frac{1}{\lambda_l^i}f(R_l^i[t])U_l^i[t]|\mathbf{Q}[t]=\mathbf{Q}\right]\mathbf{1}_{\{Q_l^i=0\}}\\
\leq&\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\frac{1}{\lambda_l^i}f(K)\leq D\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(K), \qquad (20)
\end{aligned}
$$

where $D:=\frac{1}{\min\{\lambda_l^i\}}<\infty$ and $\mathbf{1}_{\{\cdot\}}$ is the indicator function.

Next, let's focus on ΔV_2. We know that $f(R_l^i[t])=f(Q_l^i[t]+a_l^i)$ ($|a_l^i|\leq K$). According to the definition of function $f\in\mathcal{B}$, given $\epsilon>0$, there exists $M>0$, such that for any $Q_l^i[t]=Q_l^i>M$, we have $\left|\frac{f(R_l^i[t])}{f(Q_l^i)}-1\right|<\epsilon$, that is, $(1-\epsilon)f(Q_l^i)<f(R_l^i[t])<(1+\epsilon)f(Q_l^i)$. Thus, we have

$$
\begin{aligned}
&f(R_l^i[t])(A_l^i[t]-S_l^i[t])\\
=&f(R_l^i[t])\left[(A_l^i[t]-S_l^i[t])_+-(A_l^i[t]-S_l^i[t])_-\right]\\
\leq&(1+\epsilon)f(Q_l^i)(A_l^i[t]-S_l^i[t])_+-(1-\epsilon)f(Q_l^i)(A_l^i[t]-S_l^i[t])_-\\
=&f(Q_l^i)(A_l^i[t]-S_l^i[t])+\epsilon f(Q_l^i)\left|A_l^i[t]-S_l^i[t]\right|\\
\leq&f(Q_l^i)(A_l^i[t]-S_l^i[t])+K\epsilon f(Q_l^i), \qquad (21)
\end{aligned}
$$

where $(x)_+=\max\{x,0\}$, $(x)_-=-\min\{x,0\}$, and $|A_l^i[t]-S_l^i[t]|\leq|A_l^i[t]|\leq K$. Thus, we divide ΔV_2 into two parts:

$$
\begin{aligned}
\Delta V_2=&\underbrace{\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\mathbb{E}\left[\frac{1}{\lambda_l^i}f(R_l^i[t])(A_l^i[t]-S_l^i[t])|\mathbf{Q}[t]=\mathbf{Q}\right]\mathbf{1}_{\{Q_l^i>M\}}}_{=:\Delta V_3}\\
&+\underbrace{\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\mathbb{E}\left[\frac{1}{\lambda_l^i}f(R_l^i[t])(A_l^i[t]-S_l^i[t])|\mathbf{Q}[t]=\mathbf{Q}\right]\mathbf{1}_{\{Q_l^i\leq M\}}}_{=:\Delta V_4}.
\end{aligned}
$$

For ΔV_3, by using (21), we have

$$
\begin{aligned}
\Delta V_3\leq&\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\frac{1}{\lambda_l^i}f(Q_l^i)(\lambda_l^i-P_l^i)\mathbf{1}_{\{Q_l^i>M\}}\\
&+DK\epsilon\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i)\mathbf{1}_{\{Q_l^i>M\}}, \qquad (22)
\end{aligned}
$$

where $P_l^i=\mathbb{E}\left[S_l^i[t]|\mathbf{Q}[t]=\mathbf{Q}\right]=\frac{\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i)}{\sum_{k=1}^{N}\sum_{l=1}^{|\mathbf{S}^k|}f(Q_l^k)}$. Next, let's consider the term $\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\frac{1}{\lambda_l^i}f(Q_l^i)(\lambda_l^i-P_l^i)$, which can be expressed as:

$$
\begin{aligned}
&\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\frac{1}{\lambda_l^i}f(Q_l^i)(\lambda_l^i-P_l^i)\\
=&\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i)-\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\frac{f(Q_l^i)}{\lambda_l^i}\frac{\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i)}{\sum_{k=1}^{N}\sum_{l=1}^{|\mathbf{S}^k|}f(Q_l^k)}\\
=&\frac{(\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i))^2-\sum_{i=1}^{N}(\sum_{l=1}^{|\mathbf{S}^i|}\frac{f(Q_l^i)}{\lambda_l^i})(\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i))}{\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i)}.
\end{aligned}
$$

Since

$$
\sum_{i=1}^{N}(\sum_{l=1}^{|\mathbf{S}^i|}\frac{f(Q_l^i)}{\lambda_l^i})(\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i))\geq\sum_{i=1}^{N}\frac{1}{\lambda^i}(\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i))^2,
$$

where $\lambda^i=\max_{\{l=1,\ldots,|\mathbf{S}^i|\}}\lambda_l^i$, and by Lemma 1, there exists a $\delta>0$ such that

$$
\sum_{i=1}^{N}\frac{1}{\lambda^i}(\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i))^2\geq(\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i))^2(1+\delta), \qquad (23)
$$

we have

$$\sum_{i=1}^{N}(\sum_{l=1}^{|\mathbf{S}^i|}\frac{f(Q_l^i)}{\lambda_l^i})(\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i)) \geq (\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i))^2(1+\delta).$$

Thus, we get

$$\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\frac{1}{\lambda_l^i}f(Q_l^i)(\lambda_l^i - P_l^i) \leq -\delta\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i). \tag{24}$$

Hence, we have

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\frac{1}{\lambda_l^i}f(Q_l^i)(\lambda_l^i - P_l^i)\mathbf{1}_{\{Q_l^i>M\}} \\
\leq& -\delta\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i)\mathbf{1}_{\{Q_l^i>M\}} - \delta\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i)\mathbf{1}_{\{Q_l^i\leq M\}} \\
&-\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\frac{1}{\lambda_l}f(Q_l^i)(\lambda_l^i - P_l^i)\mathbf{1}_{\{Q_l^i\leq M\}} \\
\leq& -\delta\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i)\mathbf{1}_{\{Q_l^i>M\}} + \sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\frac{1}{\lambda_l^i}f(Q_l^i)P_l^i\mathbf{1}_{\{Q_l^i\leq M\}} \\
\leq& -\delta\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i)\mathbf{1}_{\{Q_l^i>M\}} + D\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(M).
\end{aligned} \tag{25}$$

Thus, we can choose ϵ small enough such that $\gamma = \delta - DK\epsilon > 0$, and thus we have

$$\begin{aligned}
\Delta V_3 &\leq -\gamma\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i)\mathbf{1}_{\{Q_l^i>M\}} + D\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(M) \\
&\leq -\gamma\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i) + (D+\gamma)\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(M)
\end{aligned}$$

For ΔV_4, we have

$$\begin{aligned}
\Delta V_4 &\leq \sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\mathbb{E}\left[\frac{1}{\lambda_l^i}f(R_l^i[t])|A_l^i[t] - S_l^i[t]||\mathbf{Q}[t] = \mathbf{Q}\right]\mathbf{1}_{\{Q_l^i\leq M\}} \\
&\leq \sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}\frac{1}{\lambda_l^i}Kf(M+K) \leq DK\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(M+K).
\end{aligned}$$

Thus, we get

$$\Delta V \leq -\gamma\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(Q_l^i) + G, \tag{26}$$

where $G := D\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(K) + DK\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(M+K) + (D+\gamma)\sum_{i=1}^{N}\sum_{l=1}^{|\mathbf{S}^i|}f(M) < \infty$.

References

[1] L. Bui, A. Eryilmaz, R. Srikant, and X. Wu. Joint asynchronous congestion control and distributed scheduling for wireless networks. In *Proc. IEEE International Conference on Computer Communications. (INFOCOM)*, Barcelona, Spain, April 2006.

[2] P. Chaporkar, K. Kar, and S. Sarkar. Throughput guarantees through maximal scheduling in wireless networks. In *Proc. Allerton Conference on Communication, Control, and Computing (Allerton)*, Monticello, Illinois, September 2005.

[3] A. Eryilmaz, A. Ozdaglar, D. Shah, and E. Modiano. Distributed cross-layer algorithms for the optimal control of multihop wireless networks. *IEEE/ACM Transactions on Networking*, 18(2):638–651, 2010.

[4] A. Eryilmaz, R. Srikant, and J. Perkins. Stable scheduling policies for fading wireless channels. *IEEE/ACM Transactions on Networking*, 13(2):411–424, 2005.

[5] L. Georgiadis, M. Neely, and L. Tassiulas. Resource allocation and cross-layer control in wireless networks. *Foundations and Trends in Networking*, 1(1):1–144, 2006.

[6] J. Ghaderi and R. Srikant. On the design of efficient csma algorithms for wireless networks, 2010. Available online at http://arxiv.org/PS_cache/arxiv/pdf/1003/1003.1364v1.pdf.

[7] L. Jiang and J. Walrand. A csma distributed algorithm for throughput and utility maximization in wireless networks. In *Proc. Allerton Conference on Communication, Control, and Computing (Allerton)*, Monticello, Illinois, September 2008.

[8] C. Joo, X. Lin, and N. Shroff. Understanding the capacity region of the greedy maximal scheduling algorithm in multi-hop wireless networks. In *Proc. IEEE International Conference on Computer Communications. (INFOCOM)*, Phoenix, Arizona, April 2008.

[9] K. Kar, S. Sarkar, and L. Tassiulas. Achieving proportional fairness using local information in aloha networks. *IEEE Transactions on Automatic Control*, 49(10):1858–1862, 2004.

[10] H. Lee, E. Modiano, and L. Le. Distributed throughput maximization in wireless networks via random power allocation. In *Proc. IEEE Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks (WiOPT)*, Seoul, Korea, June 2009.

[11] B. Li and A. Eryilmaz. A fast-csma algorithm for deadline-constrained scheduling over wireless fading channels. In *Proc. International Workshop on Resource Allocation and Cooperation in Wireless Networks (RAWNET/WNC3)*, Princeton, New Jersey, May 2011.

[12] B. Li and A. Eryilmaz. On the limitations of randomization for queue-length-based scheduling in wireless networks. In *Proc. IEEE International Conference on Computer Communications. (INFOCOM)*, Shanghai, China, April 2011.

[13] X. Lin and N. Shroff. The impact of imperfect scheduling on cross-layer rate control in multihop wireless networks. In *Proc. IEEE International Conference on Computer Communications. (INFOCOM)*, Miami, Florida, March 2005.

[14] P. Marbach, A. Eryilmaz, and A. Ozdaglar. Achievable rate region of csma schedulers in wireless networks with primary interference constraints. In *Proc. IEEE Conference on Decision and Control (CDC)*, New Orleans, Louisiana, December 2007.

[15] P. Marbach, A. Eryilmaz, and A. Ozdaglar. On the throughput-optimality of csma policies in multihop wireless networks, 2008. available at http://www.ece.osu.edu/∼eryilmaz/CSMATechReport.pdf.

[16] M. Mitzenmacher and E. Upfal. *Probability and Computing: Randomized Algorithms and Probabilistic Analysis*. Cambridge University Press, New York, NY, 2005.

[17] M. Neely. *Stochastic Network Optimization with Application to Communication and Queueing Systems*. Morgan & Claypool, 2010.

[18] M. Neely, E. Modiano, and C. Rohrs. Dynamic power allocation and routing for time varying wireless networks. In *Proc. IEEE International Conference on Computer Communications. (INFOCOM)*, San Francisco, California, April 2003.

[19] J. Ni and R. Srikant. Distributed csma/ca algorithms for achieving maximum throughput in wireless networks. In *Proc. IEEE International Workshop on Information Theory and Applications (ITA)*, San Diego, California, February 2009.

[20] S. Rajagopalan and D. Shah. Distributed algorithm and reversible network. In *Proc. IEEE Conference on Information Sciences and Systems (CISS)*, Princeton, New Jersey, March 2008.

[21] S. Rajagopalan, D. Shah, and J. Shin. Network adiabatic theorem: an efficient randomized protocol for contention resolution. In *Proc. ACM International Conference on Measurement and Modeling of Computer Systems(SIGMETRICS)*, Seattle, Washington, June 2009.

[22] S. Sanghavi, L. Bui, and R. Srikant. Distributed link scheduling with constant overhead. In *Proc. ACM International Conference on Measure-*

ment and Modeling of Computer Systems. (SIGMETRICS), San Diego, California, June 2007.

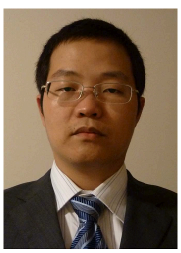
[23] G. Sasaki and B. Hajek. Link scheduling in polynomial time. *IEEE Transactions on Information Theory*, 34(5):910–917, 1988.

[24] S. Shakkottai and A. Stolyar. Scheduling for multiple flows sharing a time-varying channel: The exponential rule. *Translations of the AMS, Series 2,* A volume in memory of F. Karpelevich, 207:185–202, 2002.

[25] L. Tassiulas. Linear complexity algorithms for maximum throughput in radio networks and input queued switches. In *Proc. IEEE International Conference on Computer Communications. (INFOCOM)*, San Francisco, California, April 1998.

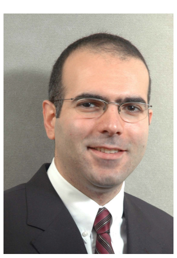
[26] L. Tassiulas and A. Ephremides. Stability properties of constrained queueing systems and scheduling policies for maximum throughput in multihop radio networks. *IEEE Transactions on Automatic Control*, 36(12):1936–1948, 1992.

[27] L. Tassiulas and A. Ephremides. Dynamic server allocation to parallel queues with randomly varying connectivity. *IEEE Transactions on Information Theory*, 39(2):466–478, 1993.



Bin Li received his B.S. degree in Electronic Engineering and M.S. degree in Communication Engineering, both from Xiamen University, China, in 2005 and 2008, respectively. Between 2008 and 2009, he was with the University of Texas at Arlington. He is currently pursuing the Ph.D. degree in the Department of Electrical and Computer Engineering at The Ohio State University.

His research interests include wireless communication and networks, optimal network control, distributed algorithms, and optimization theory.



Atilla Eryilmaz (S '00-M '06) received his B.S. degree in Electrical and Electronics Engineering from Boğaziçi University, Istanbul, in 1999, and the M.S. and Ph.D. degrees in Electrical and Computer Engineering from the University of Illinois at Urbana-Champaign in 2001 and 2005, respectively. Between 2005 and 2007, he worked as a Postdoctoral Associate at the Laboratory for Information and Decision Systems at the Massachusetts Institute of Technology. He is currently an Assistant Professor of Electrical and Computer Engineering at The Ohio State University.

Dr. Eryilmaz's research interests include design and analysis for communication networks, optimal control of stochastic networks, optimization theory, distributed algorithms, pricing in networked systems, and information theory. He received the NSF-CAREER and the Lumley Research Awards in 2010.