Cramer-Rao Lower Bound Computation Via the Characteristic Function

Steven Kay, *Fellow, IEEE*, and Cuichun Xu

Abstract

The Cramer-Rao Lower Bound is widely used in statistical signal processing as a benchmark to evaluate unbiased estimators. However, for some random variables, the probability density function has no closed analytical form. Therefore, it is very hard or impossible to evaluate the Cramer-Rao Lower Bound directly. In these cases the characteristic function may still have a closed and even simple form. In this paper, we propose a method to evaluate the Cramer-Rao Lower Bound via the characteristic function. As an example, the Cramer-Rao Lower Bound of the scale parameter and the shape parameter of the K-distribution is accurately evaluated with the proposed method. Finally it is shown that for probability density functions with a scale parameter, the Cramer-Rao Lower Bound for the remaining parameters do not depend on the scale parameter.

I. Introduction

In statistical parameter estimation theory, the Cramer-Rao Lower Bound (CRLB) is a lower bound on the variance of any unbiased estimator [1]. Let $p(x;\boldsymbol{\theta})$ denotes the probability density function (PDF) of a random variable X parameterized by $\boldsymbol{\theta}$, where $\boldsymbol{\theta} = [\theta_1, \theta_2, \cdots, \theta_L]^T$. Let $\hat{\boldsymbol{\theta}}$ be an unbiased estimator of $\boldsymbol{\theta}$. If $p(x;\boldsymbol{\theta})$ satisfies the "regularity" conditions, or

$$E\left[\frac{\partial \ln p\left(x;\boldsymbol{\theta}\right)}{\partial \boldsymbol{\theta}}\right] = \mathbf{0}$$

This work was supported by the Office of Naval Research under contracts N00014-04-10022 and N00014-04-M-0331 and by the Air Force Research Laboratory, Sensors Directorate, Hanscom AFB, MA.

Steven Kay is with the Department of Electrical Computuer and Biomedical Engineering, University of Rhode Island, Kingston, RI 02881. (Phone: 401-874-5804, Fax: 401-782-6422, e-mail: kay@ele.uri.edu)

Cuichun Xu is with the Department of Electrical Computuer and Biomedical Engineering, University of Rhode Island, Kingston, RI 02881. (Phone: 401-782-7207, Fax: 401-782-6422, e-mail: ccxu@ele.uri.edu)

where $E[\cdot]$ denotes the expectation, then

$$\text{var}\left(\hat{\theta}_i\right) \geq \left[\mathbf{I}^{-1}(\boldsymbol{\theta})\right]_{ii} = CRLB_{\theta_i} \tag{1}$$

where $\text{var}\left(\hat{\theta}_i\right)$ denotes the variance of the i^{th} element of the estimator, or the estimator for θ_i , $\left[\mathbf{I}^{-1}(\boldsymbol{\theta})\right]_{ii}$ is the (i,i) element of the inverse Fisher information matrix, and is the CRLB for θ_i. The Fisher information matrix is defined as

$$[\mathbf{I}(\boldsymbol{\theta})]_{ij} = E\left[\frac{\partial \ln p(x;\boldsymbol{\theta})}{\partial \theta_i}\frac{\partial \ln p(x;\boldsymbol{\theta})}{\partial \theta_j}\right]. \tag{2}$$

It is seen that the key step to obtain the CRLB is the evaluation of $[\mathbf{I}(\boldsymbol{\theta})]_{ij}$. Compared to other variance bounds [2], [3], the CRLB is usually easier to compute. Therefore it is extensively used in the signal processing literature as a benchmark to evaluate the performance of an unbiased estimator. However, there are cases in which $p(x;\boldsymbol{\theta})$ has no closed form. As a result, the CRLB is hard or impossible to evaluate analytically via (2). However, in some cases in which $p(x;\boldsymbol{\theta})$ has no closed form, the characteristic function (CF) of $p(x;\boldsymbol{\theta})$ does have a closed and even simple form, for example the K-distribution [4], the α-stable distribution [5] and the Class A distribution [6].

The CF of $p(x;\boldsymbol{\theta})$ is defined as

$$\phi_X(\omega;\boldsymbol{\theta}) = E\left[e^{j\omega X}\right]. \tag{3}$$

This is just the Fourier transform of $p(x;\boldsymbol{\theta})$. Therefore $p(x;\boldsymbol{\theta})$ can be thought of as a time domain function and $\phi_X(\omega;\boldsymbol{\theta})$ as a frequency domain function. Since (2) is essentially an integration in the time domain, it is possible to evaluate $\mathbf{I}(\boldsymbol{\theta})$ equivalently in the frequency domain. This idea could be applied to both univariate and multivariate distributions. In this paper, we focus on parameter CRLB evaluation of univariate distributions. The extension to multivariate distributions will be explored in the future work.

The paper is organized as follows. In Section II, the method to evaluate the CRLB via the CF is introduced. In Section III, an example is given and Section IV offers some conclusions.

II. CRLB Computation via the Characteristic Function

Equation (2) can be rewritten as

$$\begin{aligned}[\mathbf{I}(\boldsymbol{\theta})]_{ij} &= \int_{-\infty}^{\infty}\frac{\partial \ln p(x;\boldsymbol{\theta})}{\partial \theta_i}\frac{\partial \ln p(x;\boldsymbol{\theta})}{\partial \theta_j}p(x;\boldsymbol{\theta})\,dx \\ &= \int_{-\infty}^{\infty}\frac{\partial p(x;\boldsymbol{\theta})}{\partial \theta_i}\frac{\partial p(x;\boldsymbol{\theta})}{\partial \theta_j}\frac{1}{p(x;\boldsymbol{\theta})}dx. \end{aligned} \tag{4}$$

Assume there is a real positive number L, such that $p(x;\boldsymbol{\theta}) = 0$ for $|x| > \frac{L}{2}$, then

$$[\mathbf{I}(\boldsymbol{\theta})]_{ij} = \int_{-L/2}^{L/2}\frac{\partial p(x;\boldsymbol{\theta})}{\partial \theta_i}\frac{\partial p(x;\boldsymbol{\theta})}{\partial \theta_j}\frac{1}{p(x;\boldsymbol{\theta})}dx. \tag{5}$$

Let $u = \frac{x}{L}$,

$$[\mathbf{I}(\boldsymbol{\theta})]_{ij} = \int_{-1/2}^{1/2} \frac{\partial p(Lu;\boldsymbol{\theta})}{\partial \theta_i} \frac{\partial p(Lu;\boldsymbol{\theta})}{\partial \theta_j} \frac{1}{p(Lu;\boldsymbol{\theta})} L du.$$

Let $G(u;\boldsymbol{\theta}) = p(Lu;\boldsymbol{\theta})$

$$[\mathbf{I}(\boldsymbol{\theta})]_{ij} = L \int_{-1/2}^{1/2} \frac{\partial G(u;\boldsymbol{\theta})}{\partial \theta_i} \frac{\partial G(u;\boldsymbol{\theta})}{\partial \theta_j} \frac{1}{G(u;\boldsymbol{\theta})} du. \tag{6}$$

From the property of a PDF, $p(x;\boldsymbol{\theta})$ and equivalently $G(u;\boldsymbol{\theta}) \geq 0$, so we can consider $G(u;\boldsymbol{\theta})$ to be a discrete-time power spectral density (PSD). Therefore, $G(u;\boldsymbol{\theta})$ can be approximated by an autoregressive (AR) model for a sufficiently large model order, i.e. [7]

$$G(u;\boldsymbol{\theta}) \approx \frac{\sigma_u^2}{|A(u)|^2} \tag{7}$$

where σ_u^2 is the excitation noise variance, and

$$A(u) = 1 + a[1] e^{-j2\pi u} + a[2] e^{-j2\pi u 2} + \cdots + a[p] e^{-j2\pi u p}$$

where $a[1], a[2], \cdots, a[p]$ are the AR coefficients with p being the order of the AR model. Let $\mathcal{F}^{-1}$ denote the inverse Fourier transform. It is readily seen that

$$\begin{aligned} \mathcal{F}^{-1}\{A(u)\} &= \int_{-1/2}^{1/2} A(u) e^{j2\pi u n} du \\ &= a[n] \end{aligned}$$

The proper model order p can be determined by comparing the estimated AR PSD $\frac{\sigma_u^2}{|A(u)|^2}$ with the true $G(u;\boldsymbol{\theta})$ or numerically calculated $G(u;\boldsymbol{\theta})$. If the fitting error is too large, then a larger p should be used. Plugging (7) into (6), we have

$$\begin{aligned} [\mathbf{I}(\boldsymbol{\theta})]_{ij} &= \frac{L}{\sigma_u^2} \int_{-1/2}^{1/2} \left[\frac{\partial G(u;\boldsymbol{\theta})}{\partial \theta_i} A(u)\right] \left[\frac{\partial G(u;\boldsymbol{\theta})}{\partial \theta_j} A^*(u)\right] du \\ &= \frac{L}{\sigma_u^2} \int_{-1/2}^{1/2} \left[\frac{\partial G(u;\boldsymbol{\theta})}{\partial \theta_i} A(u)\right] \left[\frac{\partial G(u;\boldsymbol{\theta})}{\partial \theta_j} A(u)\right]^* du. \end{aligned} \tag{8}$$

The last equation relies on the property that $p(x;\boldsymbol{\theta})$, and therefore $G(u;\boldsymbol{\theta})$ is real. By Parseval's theorem,

$$[\mathbf{I}(\boldsymbol{\theta})]_{ij} = \frac{L}{\sigma_u^2} \sum_{n=-\infty}^{\infty} x[n] y^*[n] \tag{9}$$

where

$$x[n] = \mathcal{F}^{-1}\left\{\frac{\partial G(u,\boldsymbol{\theta})}{\partial \theta_i} A(u)\right\}$$

and

$$y[n] = \mathcal{F}^{-1}\left\{\frac{\partial G(u,\boldsymbol{\theta})}{\partial \theta_j} A(u)\right\}$$

Let $g\left[n\right]=\mathcal{F}^{-1}\left\{G\left(u;\boldsymbol{\theta}\right)\right\}$, so that $g[n]$ can be thought of as the autocorrelation sequence associated with the PSD $G\left(u;\boldsymbol{\theta}\right)$ [7]. It is given by

$$\begin{aligned} g\left[n\right] &= \int_{-1/2}^{1/2} G\left(u;\boldsymbol{\theta}\right) e^{j2\pi un} du \\ &= \int_{-1/2}^{1/2} p\left(Lu;\boldsymbol{\theta}\right) e^{j2\pi un} du. \end{aligned} \tag{10}$$

Letting $x=uL$ in the above equation yields

$$\begin{aligned} g[n] &= \int_{-1/2}^{1/2} p(x;\boldsymbol{\theta}) e^{j2\pi \frac{x}{L} n} d\frac{x}{L} \\ &= \frac{1}{L} E_X\left[e^{j2\pi\frac{n}{L}x}\right] \\ &= \frac{1}{L}\phi_X\left(\frac{2\pi n}{L};\boldsymbol{\theta}\right). \end{aligned} \tag{11}$$

Since the CF is assumed to have a closed analytical form, $g[n]$ can be easily obtained from (11). With $g[0], g[1], \cdots, g[p]$ known, the Yule-Walker method can be used to determine the AR parameters $\{a[1], a[2], \cdots, a[p], \sigma_u^2\}$ [7].

Next, since the inverse Fourier transform is linear, we have

$$\mathcal{F}^{-1}\left\{\frac{\partial G\left(u;\boldsymbol{\theta}\right)}{\partial\theta_i}\right\} = \frac{\partial \mathcal{F}^{-1}\left\{G\left(u;\boldsymbol{\theta}\right)\right\}}{\partial\theta_i}. \tag{12}$$

Plugging (11) into (12), we have

$$\begin{aligned} \mathcal{F}^{-1}\left\{\frac{\partial G\left(u;\boldsymbol{\theta}\right)}{\partial\theta_i}\right\} &= \frac{\partial \frac{1}{L}\phi_X\left(\frac{2\pi n}{L};\boldsymbol{\theta}\right)}{\partial\theta_i} \\ &= \frac{1}{L}\frac{\partial}{\partial\theta_i}\phi_X\left(\frac{2\pi n}{L};\boldsymbol{\theta}\right). \end{aligned} \tag{13}$$

Let

$$h_i[n] = \frac{\partial}{\partial\theta_i}\phi_X\left(\frac{2\pi n}{L};\boldsymbol{\theta}\right) \tag{14}$$

so that from (9)

$$\left[\mathbf{I}\left(\boldsymbol{\theta}\right)\right]_{ij} = \frac{1}{L\sigma_u^2}\sum_{n=-\infty}^{\infty}\left(a\left[n\right]\star h_i\left[n\right]\right)\left(a\left[n\right]\star h_j\left[n\right]\right)^* \tag{15}$$

where $\star$ denotes the convolution.

If there is a real positive integer M, such that $h_i[n]\approx 0$, and $h_j[n]\approx 0$ for $|n|>M$, then (15) can be approximated as

$$\left[\mathbf{I}\left(\boldsymbol{\theta}\right)\right]_{ij} = \frac{1}{L\sigma_u^2}\sum_{n=-M}^{M}\left(a\left[n\right]* h_i\left[n\right]\right)\left(a\left[n\right]* h_j\left[n\right]\right)^*. \tag{16}$$

After the analytical form of $h_i[n]$ and $h_j[n]$ are obtained, M can be obtained.

For an arbitrary $p(x;\boldsymbol{\theta})$, we can always find the positive number L, such that $p(x;\boldsymbol{\theta}) \approx 0$ for $|x| > \frac{L}{2}$. This may be done directly from the expression of $p(x;\boldsymbol{\theta})$. However, since we are considering the case when $p(x;\boldsymbol{\theta})$ has no closed form, it is necessary to be able to determine L from $\phi_X(\omega;\boldsymbol{\theta})$. To do so note that the moments of X can be obtained by [8]

$$E[X^n] = \frac{1}{j^n} \frac{d^n \phi_X(\omega)}{d\omega^n}\bigg|_{\omega=0}.$$

From the moments of X and $\phi_X(\omega;\boldsymbol{\theta})$, L can be found using probability bounds. In the example of the next section, the Chernoff bound is used to determine L [9]. The steps to evaluate $[\mathbf{I}(\boldsymbol{\theta})]_{ij}$ via the CF $\phi_X(\omega;\boldsymbol{\theta})$ are summarized below,

1) From $\phi_X(\omega;\boldsymbol{\theta})$, obtain L such that $p(x;\boldsymbol{\theta}) \approx 0$ for $|x| > \frac{L}{2}$, by using a bound.
2) Select an AR model order p.
3) Calculate $g[0], g[1], \cdots, g[p]$ from (11).
4) Determine $a[1], a[2], \cdots a[p], \sigma_u^2$ by the Yule-Walker method.
5) Obtain the expression of $h_i[n]$ and $h_j[n]$ from (14).
6) Select M from the expression of $h_i[n]$ and $h_j[n]$.
7) Evaluate $\mathbf{I}(\boldsymbol{\theta})$ from (16).

We note that in the above procedure the PDF $p(x;\boldsymbol{\theta})$ *need not be known.*

III. CRLB Computation for the K-distribution

The K-distribution is widely used to model sea surface clutter in high-resolution radar systems and to model sea floor reverberation in high-resolution active sonar systems [4], [10], [11], [12].

Let $X = X_r + jX_i$ be a zero mean, circularly symmetric complex Gaussian random variable, i.e. $[X_r, X_i]^T \sim N(\mathbf{0}, \sigma^2\mathbf{I})$, where $\mathbf{I}$ is a 2×2 identity matrix. Let Y be a Gamma random variable with the shape parameter being $\nu + 1$ and the scale parameter being $1/2$, or

$$p_Y(y) = \begin{cases} \frac{\left(\frac{1}{2}\right)^{\nu+1}}{\Gamma(\nu+1)} y^{\nu} \exp\left(-\frac{1}{2}y\right) & y \geq 0 \\ 0 & y < 0 \end{cases} \quad (17)$$

where $\Gamma(\cdot)$ denotes the Gamma function and $\nu > -1$. Then $\sqrt{Y}X$ is distributed according to a complex K-distribution [12]. Denote this variable as $R = Z + jT$, where Z is the real part of R and T is the imaginary part. The joint CF of (Z, T) is given by [10]

$$\begin{aligned} \phi_{Z,T}(\omega, \gamma) &= E\left[e^{j\omega Z + j\gamma T}\right] \\ &= \frac{1}{\left(1 + \sigma^2(\omega^2 + \gamma^2)\right)^{\nu+1}}. \end{aligned} \quad (18)$$

Hence, the CF of Z is given by

$$\phi_Z(\omega) = \phi_{Z,T}(\omega, 0) = \frac{1}{(1+\sigma^2\omega^2)^{\nu+1}} \tag{19}$$

and the PDF of Z can be shown to be

$$p_Z(z) = \frac{1}{\sqrt{\pi\sigma^2}\Gamma(\nu+1)} \left(\frac{|z|}{2\sigma}\right)^{\nu+1/2} K_{\nu+1/2}\left(\frac{|z|}{\sigma}\right) \tag{20}$$

where $K_\alpha(\cdot)$ is the modified Bessel function of the second kind.

From (20) it is seen that σ is the scale parameter and ν is the shape parameter. The moments of Z can be obtained from the CF as

$$E[Z] = \frac{1}{j}\frac{\partial\phi_Z(\omega)}{\partial\omega}\bigg|_{\omega=0} = 0 \tag{21}$$

and

$$E\left[Z^2\right] = -\frac{\partial^2\phi_Z(\omega)}{\partial\omega^2}\bigg|_{\omega=0} = 2(\nu+1)\sigma^2. \tag{22}$$

Therefore the variance of Z is

$$\text{var}(Z) = E\left[Z^2\right] - E^2[Z] = 2(\nu+1)\sigma^2.$$

It is seen that as ν decreases, the variance of Z decreases. When $\nu \to -1$, $\text{var}(Z) \to 0$, which means Z approaches the deterministic value 0.

In [10] an approximate CRLB for ν of the PDF of $|R|$ is derived under the condition that σ^2 is known and $\nu \gg 0$. In the rest of this paper, we will focus on computing the CRLBs for both ν and σ^2 of $p_Z(z)$ without any additional constrain on the range of ν. Considering the expression of (20), it would be a formidable task to evaluate $[\mathbf{I}(\boldsymbol{\theta})]_{ij}$ using (2). Therefore we evaluate $[\mathbf{I}(\boldsymbol{\theta})]_{ij}$ via the CF using the method introduced in the last section.

A. *Selection of* L

L is determined from the Chernoff bound, which is given as [9]

$$\begin{aligned} prob(z \geq \tau) &\leq \exp(-s\tau)E[\exp(sz)] \\ &= \exp(-s\tau + \log(E[\exp(sz)])) \end{aligned} \tag{23}$$

for any $s \geq 0$. With this bound, an L is found as

$$L = \begin{cases} 2\sqrt{15}(\nu+1)\sigma & \text{for} \quad \nu \geq 4 \\ 10\sqrt{15}\sigma & \text{for} \quad \nu < 4 \end{cases} \tag{24}$$

The details are given in Appendix I.

In general, a simple procedure to determine L is to increase it, until the results converge.

B. Selection of the AR Model Order p

It can be seen that from (20) as $\nu \to -1$, $p_Z(z)$ becomes narrower, requiring p to be large. The proper value of p is also determined by L. This is because as L increases, the PSD $G(u;\boldsymbol{\theta})$ becomes narrower, requiring a larger p. For the selection of L given by (24), an empirical formula is obtained from simulations, and is given as

$$p = \begin{cases} \lceil 1200 \exp(-7\nu) + 70 \rceil & \text{for} \quad -1 < \nu < 0.9 \\ 50 & \text{for} \quad \nu \geq 0.9 \end{cases}$$

where $\lceil \cdot \rceil$ denotes "the smallest integer greater than".

In general, a simple procedure to determine p is to increase it, until the results converge.

C. Expression of $g[n]$ and $h_i[n]$

Plugging (19) into (11) we have

$$g[n] = \left. \frac{1}{L(1+\sigma^2\omega^2)^{\nu+1}} \right|_{\omega = \frac{2\pi n}{L}}$$

Plugging (19) into (14) yields

$$\begin{aligned} h_{\sigma^2}[n] &= \left. \frac{\partial}{\partial \sigma^2} \left[\frac{1}{(1+\sigma^2\omega^2)^{\nu+1}} \right] \right|_{\omega = \frac{2\pi n}{L}} \\ &= \left. \frac{\omega^2(1+\nu)}{(1+\sigma^2\omega^2)^{\nu+2}} \right|_{\omega = \frac{2\pi n}{L}} \end{aligned} \tag{25}$$

and

$$\begin{aligned} h_{\nu}[n] &= \left. \frac{\partial}{\partial \nu} \left[\frac{1}{(1+\sigma^2\omega^2)^{\nu+1}} \right] \right|_{\omega = \frac{2\pi n}{L}} \\ &= \left. \frac{\ln(1+\sigma^2\omega^2)}{(1+\sigma^2\omega^2)^{\nu+1}} \right|_{\omega = \frac{2\pi n}{L}} \end{aligned} \tag{26}$$

D. Selection of M

From the derivation in Appendix II,

$$M = \min(M_1, M_2)$$

where

$$M_1 = \left\lceil \frac{L}{2\pi} \left(\frac{1+\nu}{10^{-4}} \right)^{\frac{1}{2\nu+2}} \sigma^{-\frac{2\nu+4}{2\nu+2}} \right\rceil .$$

and

$$M_2 = \begin{cases} \min\left(\left\lceil \frac{L}{2\pi\sigma} 341.6 \right\rceil, \left\lceil \frac{L}{2\pi\sigma} \left(\frac{1}{10^{-4}}\right)^{\frac{2}{2\nu+1}} \right\rceil\right) & \nu \geq 0 \\ \left\lceil \frac{L}{2\pi\sigma} \sqrt{x_\nu} \right\rceil & -1 < \nu < 0 \end{cases}$$

where x_ν is the greatest root of the equation

$$\frac{\log(1+x)}{(1+x)^{1+\nu}} = 10^{-4}$$

and it can be easily found by a numerical method.

In general, a simple procedure to determine M is to increase it, until the results converge.

E. A Corroboration Test

Now we have everything to calculate $\mathbf{I}(\boldsymbol{\theta})$ from (16). As a corroboration test of the proposed method, a special case in which $\nu = 0$ is considered first. When $\nu = 0$, it is readily shown that

$$p_Z(z) = \frac{1}{2\sigma} \exp\left(-\frac{|z|}{\sigma}\right). \tag{27}$$

After some simple algebra, we have

$$[\mathbf{I}(\boldsymbol{\theta})]_{\sigma^2\sigma^2} = E\left[\left(\frac{\partial \ln p(x;\sigma^2)}{\partial \sigma^2}\right)^2\right] = \frac{1}{4\sigma^4}. \tag{28}$$

The theoretical value given by (28) and the value computed from the proposed method are shown in Fig. 1 as circles and stars respectively. It can be seen that the match is very good and the results are precise enough for most applications. However, if higher precision is required, we can always increase L, p and M.

F. CRLB Evaluation of ν and σ^2 for the K-PDF of (20)

Even though there are two parameters in $p_Z(z)$, it is shown in Appendix III that the $CRLB_\nu$ is independent of σ^2, and for a fixed ν, $CRLB_{\sigma^2}$ is a linear function of σ^4. This is because σ is a scale parameter of $p_Z(z)$. Therefore, there is no need to plot the 2 dimensional figures of $CRLB_{\sigma^2}$ and $CRLB_\nu$ versus σ^2 and ν. In Fig. 2 the computed $CRLB_\nu$ is plotted versus ν and in Fig. 3 the computed $\frac{CRLB_{\sigma^2}}{\sigma^4}$ is plotted versus ν. In both figures the computed values are connected with straight line for better viewing.

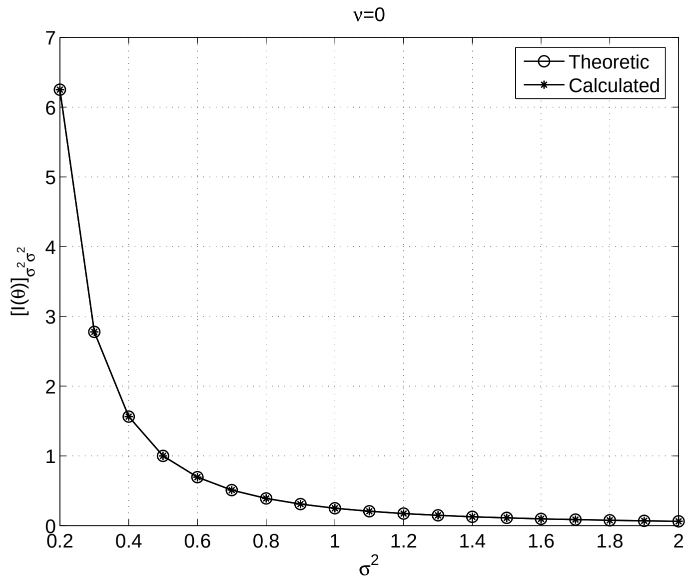

Fig. 1. Theoretical and computed $[\mathbf{I}(\boldsymbol{\theta})]_{\sigma^2\sigma^2}$ versus σ^2 for the K-PDF of (20) with $\nu = 0$.

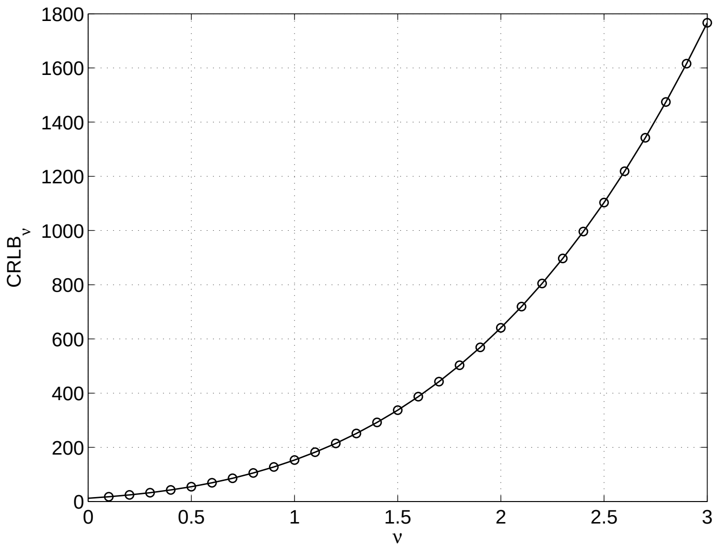

Fig. 2. Evaluated $CRLB_\nu$ versus ν for all σ^2

.

IV. CONCLUSION

We have proposed a method to evaluate the CRLB via the CF. When the PDF has no closed form, it can be very hard or impossible to evaluate the CRLB directly. With the proposed method, the CRLB

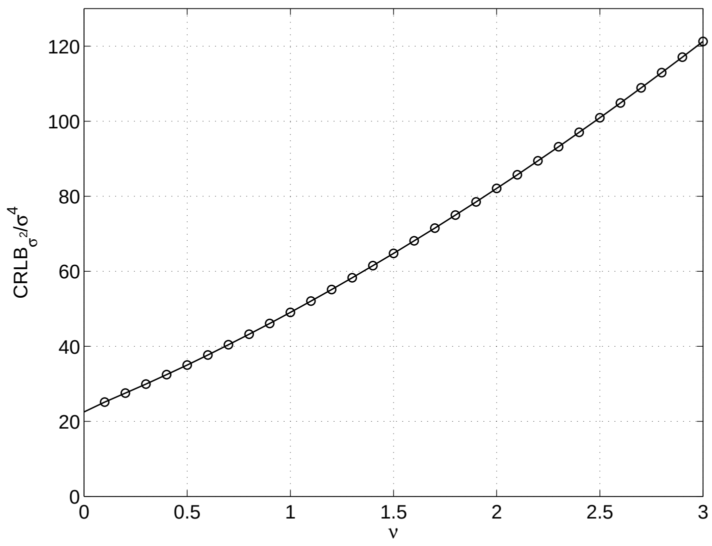

Fig. 3. Evaluated $\frac{CRLB_{\sigma^2}}{\sigma^4}$ versus ν

.

can be accurately evaluated when the CF is known. As an example, the CRLBs of the scale parameter and the shape parameter of the K-distribution have been computed with the proposed approach. It has also been shown that for a PDF with a scale parameter, the CRLB for the remaining parameters do not depend on the scale parameter.

APPENDIX I
SELECTION OF L

From (19), the moment generation function of Z is given as

$$E\left[\zeta^{z}\right]=\frac{1}{\left(1-\sigma^{2}\zeta^{2}\right)^{\upsilon+1}}.$$

Letting $\zeta=s$ and minimizing logarithm of (23)

$$F\left(s\right)=-s\tau-\left(\nu+1\right)\log\left(1-\sigma^{2}s^{2}\right)$$

with respect to s, we have

$$F\left(\hat{s}\right)=-\sqrt{\left(\nu+1\right)^{2}+\frac{\tau^{2}}{\sigma^{2}}}+\nu+1-\left(\nu+1\right)\log\left(1-\frac{\left(\sqrt{\sigma^{2}\left(\nu+1\right)^{2}+\tau^{2}}-\sigma\left(\nu+1\right)^{2}\right)^{2}}{\tau^{2}}\right).$$

If we let $\tau = \sqrt{15\,(\nu+1)^2\,\sigma^2}$ then

$$F\left(\hat{s}\right) = -3 - \log\left(\frac{2}{5}\right) - \left(3 + \log\left(\frac{2}{5}\right)\right)\nu.$$

It can be shown that for $\nu < 4$

$$prob\left(\left.z \geq \tau\right|\nu < 4\right) \leq prob\left(\left.z \geq \tau\right|\nu = 4\right) \leq \exp\left(F\left(\hat{s};\nu = 4\right)\right) < 2.987 \times 10^{-5}.$$

Hence if we choose

$$\tau = \begin{cases} \sqrt{15}\,(\nu+1)\,\sigma & \text{for} \quad \nu \geq 4 \\ 5\sqrt{15}\sigma & \text{for} \quad \nu < 4 \end{cases} \tag{29}$$

then $prob\left(z \geq \tau\right) < 2.987 \times 10^{-5}$. Since $p_Z\left(z\right)$ is symmetric we have

$$prob\left(z \leq -\tau\right) < 2.987 \times 10^{-5}.$$

Letting $L = 2\tau$, then $p_Z\left(z\right) \approx 0$ for $|z| > \frac{L}{2}$.

APPENDIX II
SELECTION OF M

Differentiating (19) with respect to σ^2 and taking the absolute value yield

$$\begin{aligned} \left|\frac{d}{d\sigma^2}\phi_X\left(\omega\right)\right| &= \frac{\omega^2\left(1+\nu\right)}{\left(1+\sigma^2\omega^2\right)^{\nu+2}} \\ &< \frac{\omega^2\left(1+\nu\right)}{\left(\sigma^2\omega^2\right)^{\nu+2}} \\ &= \frac{\left(1+\nu\right)}{\left(\sigma^2\right)^{\nu+2}\omega^{2\nu+2}}. \end{aligned} \tag{30}$$

Hence, we have $\left|\frac{d}{d\sigma^2}\phi_X\left(\omega\right)\right| < 10^{-4}$, if

$$\omega > \left(\frac{1+\nu}{\sigma^{2\nu+4}10^{-4}}\right)^{\frac{1}{2\nu+2}} = \left(\frac{1+\nu}{10^{-4}}\right)^{\frac{1}{2\nu+2}}\sigma^{-\frac{2\nu+4}{2\nu+2}}.$$

Equivalently, $\left|h_{\sigma^2}\left[n\right]\right| < 10^{-4}$ for all $n \geq M_1$, if

$$M_1 = \left\lceil \frac{L}{2\pi}\left(\frac{1+\nu}{10^{-4}}\right)^{\frac{1}{2\nu+2}}\sigma^{-\frac{2\nu+4}{2\nu+2}}\right\rceil.$$

Differentiating (19) with respect to ν and taking the absolute value yield

$$\begin{aligned} \left|\frac{d}{d\nu}\phi_X(\omega)\right| &= \frac{\ln\left(1+\sigma^2\omega^2\right)}{\left(1+\sigma^2\omega^2\right)^{\nu+1}} \\ &< \frac{\ln\left(1+\sigma^2\omega^2\right)}{\left(\sigma^2\omega^2\right)^{\nu+1}} \\ &\le \frac{\left(\sigma^2\omega^2\right)^{1/2}}{\left(\sigma^2\right)^{\nu+2}\omega^{2\nu+2}} \\ &= \frac{1}{\left(\sigma^2\omega^2\right)^{\nu+1/2}}. \end{aligned} \tag{31}$$

If $\nu > -\frac{1}{2}$ then $\left|\frac{d}{d\nu}\phi_X(\omega)\right| < 10^{-4}$ for

$$\omega \ge \frac{1}{\sigma}\left(\frac{1}{10^{-4}}\right)^{\frac{2}{2\nu+1}}.$$

Or equivalently $|h_\nu[n]| < 10^{-4}$ for all $n \ge M_2$, if

$$M_2 \ge \left\lceil \frac{L}{2\pi\sigma}\left(\frac{1}{10^{-4}}\right)^{\frac{2}{2\nu+1}} \right\rceil.$$

However, when $\nu \to 0$, M_2 could be very large and this bound is not tight enough. Therefore, another bound is derived in the follows.

For $\nu > 0$

$$\frac{\log\left(1+\sigma^2\omega^2\right)}{\left(1+\sigma^2\omega^2\right)^{\nu+1}} < \frac{\log\left(1+\sigma^2\omega^2\right)}{\left(1+\sigma^2\omega^2\right)}$$

solving the equation $\frac{\log(1+x)}{1+x} = 10^{-4}$ we have $x = 1.1667 \times 10^5$, so

$$\frac{\log\left(1+\sigma^2\omega^2\right)}{\left(1+\sigma^2\omega^2\right)^{\nu+1}} < 10^{-4}$$

if $\omega \ge \frac{1}{\sigma}341.6$. Or equivalently $|h_\nu[n]| < 10^{-4}$ for all $n \ge M_2$, if

$$M_2 = \left\lceil \frac{L}{2\pi\sigma}341.6 \right\rceil.$$

Therefore for $\nu > 0$ we have

$$M_2 = \min\left(\left\lceil \frac{L}{2\pi\sigma}341.6 \right\rceil, \left\lceil \frac{L}{2\pi\sigma}\left(\frac{1}{10^{-4}}\right)^{\frac{2}{2\nu+1}} \right\rceil\right).$$

For $-1 < \nu \le 0$, the equation

$$\frac{\log(1+x)}{(1+x)^{1+\nu}} = 10^{-4}$$

is first solved numerically. Let x_ν be the greatest root, then

$$\frac{\log\left(1+\sigma^2\omega^2\right)}{\left(1+\sigma^2\omega^2\right)^{\nu+1}} < 10^{-4}$$

if $\omega > \frac{\sqrt{x_\nu}}{\sigma}$. Or equivalently $|h_\nu[n]| < 10^{-4}$ for all $n \geq M_2$, if

$$M_2 = \left\lceil \frac{L}{2\pi\sigma}\sqrt{x_\nu} \right\rceil .$$

So finally,

$$M_2 = \begin{cases} \min\left(\left\lceil \frac{L}{2\pi\sigma} 341.6 \right\rceil , \left\lceil \frac{L}{2\pi\sigma} \left(\frac{1}{10^{-4}} \right)^{\frac{2}{2\nu+1}} \right\rceil \right) & \nu \geq 0 \\ \left\lceil \frac{L}{2\pi\sigma}\sqrt{x_\nu} \right\rceil & -1<\nu < 0 \end{cases}$$

and

$$M = \min\left(M_1, M_2\right).$$

Appendix III
Independence of the CRLB of a Scale Parameter

Theorem 1: Let $p_Z(z;\alpha,\lambda)$ be a PDF parameterized by α and λ. If $p_Z(z;\alpha,\lambda)$ satisfies the "regularity condition" and λ is a positive scale parameter, or $p_Z(z;\alpha,\lambda) = \lambda f(\lambda z;\alpha)$, then $CRLB_\alpha$ is independent of the value of λ, and for a fixed α, $CRLB_\lambda$ is a linear function of λ^2.

We note that the CRLB for α will be smaller when λ is known. Also, in the *Theorem 1* there is only one remaining parameter, i.e. α. When there are more than one remaining parameters the same conclusion still holds, and it can be proved similarly.

Proof: By definition, the $(1,1)$ element of the Fisher information matrix is

$$\begin{aligned} \mathbf{I}_{\alpha\alpha} &= E\left[\left(\frac{\partial \ln\left(p_Z\left(z;\alpha,\lambda\right)\right)}{\partial\alpha} \right)^2 \right] \\ &= \int_{-\infty}^{\infty} \left(\frac{\partial\left(p_Z\left(z;\alpha,\lambda\right)\right)}{\partial\alpha} \right)^2 \frac{1}{p_Z\left(z;\alpha,\lambda\right)} dz \\ &= \int_{-\infty}^{\infty} \left(\frac{\partial \lambda f\left(\lambda z;\alpha\right)}{\partial\alpha} \right)^2 \frac{1}{\lambda f\left(\lambda z;\alpha\right)} dz. \end{aligned} \tag{32}$$

It is assumed that the range of the PDF is from $-\infty$ to ∞. For PDFs with other ranges, the theorem can be proved similarly. In the rest of the proof the limits of the integral are omitted for simplicity. Let $x = \lambda z$, after straightforward algebra, we have from (32)

$$\mathbf{I}_{\alpha\alpha} = \int \left(\frac{\partial f\left(x;\alpha\right)}{\partial\alpha} \right)^2 \frac{1}{f\left(x;\alpha\right)} dx. \tag{33}$$

This is a function of α only, so it can be denoted as $g_1(\alpha)$.

By definition

$$\begin{aligned}\mathbf{I}_{\lambda\lambda} &= \int \left(\frac{\partial\left(p_Z\left(z;\alpha,\lambda\right)\right)}{\partial\lambda}\right)^2 \frac{1}{p_Z\left(z;\alpha,\lambda\right)}dz \\ &= \int \left(\frac{\partial\lambda f\left(\lambda z;\alpha\right)}{\partial\lambda}\right)^2 \frac{1}{\lambda f\left(\lambda z;\alpha\right)}dz \\ &= \int \left(f\left(\lambda z;\alpha\right) + \lambda z \left.\frac{\partial f\left(u;\alpha\right)}{\partial u}\right|_{u=\lambda z}\right)^2 \frac{1}{\lambda f\left(\lambda z;\alpha\right)}dz. \end{aligned} \tag{34}$$

Let $x = \lambda z$, after straightforward algebra, we have from (34)

$$\mathbf{I}_{\lambda\lambda} = \int \left(f\left(x;\alpha\right) + x \left.\frac{\partial f\left(u;\alpha\right)}{\partial u}\right|_{u=x}\right)^2 \frac{1}{\lambda^2 f\left(x;\alpha\right)}dx. \tag{35}$$

This can be denoted as $\frac{1}{\lambda^2}g_2(\alpha)$, where $g_2(\alpha)$ is a function of α only.

Similarly

$$\begin{aligned}\mathbf{I}_{\alpha\lambda} &= \int \left(\frac{\partial\left(p_Z\left(z;\alpha,\lambda\right)\right)}{\partial\alpha}\right)\left(\frac{\partial\left(p_Z\left(z;\alpha,\lambda\right)\right)}{\partial\lambda}\right) \frac{1}{p_Z\left(z;\alpha,\lambda\right)}dz \\ &= \int \left(\frac{\partial\left(\lambda f\left(\lambda z;\alpha\right)\right)}{\partial\alpha}\right)\left(f\left(\lambda z;\alpha\right) + \lambda z \left.\frac{\partial f\left(u;\alpha\right)}{\partial u}\right|_{u=\lambda z}\right) \frac{1}{\lambda f\left(\lambda z;\alpha\right)}dz. \end{aligned} \tag{36}$$

Let $x = \lambda z$, after straightforward algebra, we have from (36)

$$\mathbf{I}_{\alpha\lambda} = \int \left(\frac{\partial\left(f\left(x;\alpha\right)\right)}{\partial\alpha}\right)\left(f\left(x;\alpha\right) + x \left.\frac{\partial f\left(u;\alpha\right)}{\partial u}\right|_{u=x}\right) \frac{1}{\lambda f\left(x;\alpha\right)}dx. \tag{37}$$

This can be denoted as $\frac{1}{\lambda}g_3(\alpha)$, where $g_3(\alpha)$ is a function of α only.

Therefore, the Fisher information matrix can be expressed as

$$\mathbf{I}\left(\alpha,\lambda\right) = \begin{bmatrix} g_1\left(\alpha\right) & \frac{1}{\lambda}g_3\left(\alpha\right) \\ \frac{1}{\lambda}g_3\left(\alpha\right) & \frac{1}{\lambda^2}g_2\left(\alpha\right) \end{bmatrix}.$$

As a result,

$$\mathbf{I}^{-1}\left(\alpha,\lambda\right) = \frac{1}{g_1\left(\alpha\right)g_2\left(\alpha\right) - g_3^2\left(\alpha\right)} \begin{bmatrix} g_2\left(\alpha\right) & -\lambda g_3\left(\alpha\right) \\ -\lambda g_3\left(\alpha\right) & \lambda^2 g_1\left(\alpha\right) \end{bmatrix}.$$

This expression proves the theorem. ■

Considering the K-distribution PDF given by (20), according to *Theorem 1*, $CRLB_\nu$ is independent of $\frac{1}{\sigma}$. Hence $CRLB_\nu$ is independent of σ^2. Also $CRLB_{\frac{1}{\sigma}}$ is a linear function of $\frac{1}{\sigma^2}$. After parameter transformation it can be shown [1], $CRLB_{\sigma^2}$ is a linear function of σ^4.

References

[1] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*, Englewood Cliffs, NJ: Prentice-Hall, 1993.

[2] R. J. McAulay and E. M. Hofstetter, “Barankin Bounds on Parameter Estimation,” *IEEE Trans. Inform. Theory*, vol. 17, pp. 669–676, Nov. 1971.

[3] M. G. Kendall and A. Stuart, *The Advanced Theory of Statistics*, vol. 2, Macmillan, New York: 1979.

[4] E. Jakeman and P. N. Pusey, “A Model for Non-Rayleigh Sea Echo,” *IEEE Trans. Antennas Propagation*, vol. Ap-24, pp. 806–814, Nov. 1976.

[5] M. Shao and C. L. Nikias, “Signal processing with fractional lower order moments: stable processes and their applications,” *Proc. IEEE*, vol. 81, pp. 986–1010, July. 1993.

[6] D. Middleton, “Canonical and Quasi-Canonical Probability Models of Class A Interference,” *IEEE Trans. Electromagnetic Compat.*, EMC-25, pp. 76–106, May. 1983.

[7] S. M. Kay, *Modern Spectral Estimation: Theory and Application*, Englewood Cliffs, NJ: Prentice-Hall, 1988.

[8] S. M. Kay, *Intuitive Probability and Random Processes using MATLAB*, New York: Springer, 2006.

[9] H. L. V. Trees, *Detection Estimation and Modulation Theory, Part I*, New York: Wiley, 1968.

[10] D. A. Abraham and A. P. Lyons, “Novel physical interpretations of K-distributed reverberation,” *IEEE Journal of Oceanic Engineering*, vol. 27, no.4, pp. 800–813, Oct. 2002.

[11] A. P. Lyons and D. A. Abraham, “Statistical characterization of high frequency shallow-water sea-floor backscatter,” *J. Acoust. Soc. Amer.,* vol. 106, no 3, pp. 1307–1315, 1999.

[12] K. D. Ward, “Compound representation of high resolution sea clutter,” *Electron. Lett.*, vol. 17, no.16, pp. 561–563, Oct. 2002.