Rapid Estimation of the Parameters of a K-Distribution

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Abstract

1 Introduction

The K-PDF is a good model for clutter when the scatterers are not homogeneous. The random variable that describes a noise sample is given by

$$Z = \sqrt{V}U$$

where V is a Gamma distributed random variable with shape parameter $\alpha = \nu + 1$, and scale parameter $\lambda = 1/2$, and U is Gaussian with mean zero and variance σ^2. The random variables U and V are independent. The PDF can be shown to be

$$p_Z(z) = \frac{1}{\sqrt{\pi\sigma^2}\Gamma(\nu+1)} \left(\frac{|z|}{2\sigma}\right)^{\nu+1/2} K_{\nu+1/2}\left(\frac{|z|}{\sigma}\right) \qquad -\infty < z < \infty. \tag{1}$$

In this paper we examine the use of a simple but effective estimation procedure for the parameters ν and σ^2. Although the maximum likelihood estimator is the asymptotically optimal estimator, it is difficult to implement it for the K-PDF. As an alternative estimator, we use the cumulant generating function (CGF) approach as previously described in [2].

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2 Estimation Method

It can easily be shown that the characteristic function for the PDF of (**??**) is

$$\phi_Z(\omega) = E\left(\exp(j\omega Z)\right) = \frac{1}{(1+\omega^2\sigma^2)^{\nu+1}} \tag{2}$$

for all ω. As a result the CGF is

$$\begin{aligned} K_Z(\omega) &= \log\left[\phi_Z(\omega)\right] \\ &= -(\nu+1)\log\left(1+\sigma^2\omega^2\right). \end{aligned} \tag{3}$$

We see that the CGF is now linear in the unknown shape parameter ν and nonlinear in the scale parameter σ^2. Hence, we can estimate it by fitting the *estimated CGF* in a least squares sense. The estimated CGF is defined as

$$\hat{K}_Z(\omega) = \log\left(\frac{1}{N}\sum_{i=1}^{N}\cos(\omega z_i)\right) \tag{4}$$

where it is assumed that N independent and identically distributed (IID) samples of Z or $\{z_1, z_2, \ldots, z_N\}$ have been observed. Also, note that the K-PDF is an even function so that its characteristic function and hence its CGF is real. To estimate ν we minimize the least squares error

$$J(\nu,\sigma^2) = \sum_{k=0}^{M}\left(\hat{K}_Z(\omega_k) - K_Z(\omega_k)\right)^2 \tag{5}$$

over a suitable range of ω's, $\omega_0 \le \omega \le \omega_M$. To do so we first minimize with respect to the linear parameter ν of

$$\begin{aligned} J(\theta_1,\theta_2) &= \sum_{k=0}^{M}\left[\hat{K}_Z(\omega_k) - \left(\underbrace{-(\nu+1)}_{\theta_1}\log\left(1+\underbrace{\sigma^2}_{\theta_2}\omega_k^2\right)\right)\right]^2 \\ &= \sum_{k=0}^{M}\left(\hat{K}_Z(\omega_k) - \theta_1\log\left(1+\theta_2\omega_k^2\right)\right)^2 \\ &= (\hat{\mathbf{K}} - \mathbf{H}(\theta_2)\theta_1)^T(\hat{\mathbf{K}} - \mathbf{H}(\theta_2)\theta_1) \end{aligned} \tag{6}$$

where we have let

$$\hat{\mathbf{K}} = \begin{bmatrix} \hat{K}_Z(\omega_0) \\ \hat{K}_Z(\omega_1) \\ \vdots \\ \hat{K}_Z(\omega_M) \end{bmatrix} \qquad \mathbf{H}(\theta_2) = \begin{bmatrix} \ln(1+\theta_2\omega_0^2) \\ \ln(1+\theta_2\omega_1^2) \\ \vdots \\ \ln(1+\theta_2\omega_M^2) \end{bmatrix}.$$

The minimization of $J(\theta_1,\theta_2)$ over θ_1 produces the result [1]

$$\hat{\theta_1} = \left(\mathbf{H}^T(\theta_2)\mathbf{H}(\theta_2)\right)^{-1}\mathbf{H}^T(\theta_2)\hat{\mathbf{K}} \tag{7}$$

which when substituted into (??) produces

$$J(\hat{\theta}_1, \theta_2) = \hat{\mathbf{K}}^T \left(\mathbf{I} - \mathbf{H}(\theta_2)\left(\mathbf{H}^T(\theta_2)\mathbf{H}(\theta_2)\right)^{-1}\mathbf{H}^T(\theta_2)\right)\hat{\mathbf{K}}$$

so that equivalently we need to *maximize*

$$L(\theta_2) = \hat{\mathbf{K}}^T\mathbf{H}(\theta_2)\left(\mathbf{H}^T(\theta_2)\mathbf{H}(\theta_2)\right)^{-1}\mathbf{H}^T(\theta_2)\hat{\mathbf{K}}$$

over θ_2. This can be written as

$$L(\theta_2) = \frac{\left[\sum_{k=0}^{M} \hat{K}_Z(\omega_k)\ln(1+\theta_2\omega_k^2)\right]^2}{\sum_{k=0}^{M}\left[\ln(1+\theta_2\omega_k^2)\right]^2} \tag{8}$$

and must be maximized over $\theta_2 > 0$ using a grid search. Once the maximizing value of $\theta_2 = \sigma^2$ has been found, then this value is the least squares estimate $\hat{\theta}_2 = \hat{\sigma^2}$. With this value, the estimate of θ_1 easily follows from (??) as

$$\hat{\theta}_1 = \frac{\sum_{k=0}^{M} \hat{K}_Z(\omega_k)\ln(1+\hat{\theta}_2\omega_k^2)}{\sum_{k=0}^{M}\left[\ln(1+\hat{\theta}_2\omega_k^2)\right]^2} \tag{9}$$

and thus, $\hat{\nu} = -(1+\hat{\theta}_1)$. The only parameter that needs to be specified is the set of ω's for which the error is minimized over. Typically, we choose these to be equally spaced over an interval $[0, 0.5]$.

3 Computer Simulation Results

The estimator described for ν and σ^2 was implemented to determine its bias and variance. As a basis for comparison we also compare the performance to that of a method of moments estimator. The latter is easy to implement but has no optimality properties [1]. It is easily shown that

$$\begin{aligned} E[Z^2] &= 2(\nu+1)\sigma^2 \\ E[Z^4] &= 12(\nu+1)(\nu+2)(\sigma^2)^2 \end{aligned}$$

so that solving for the unknown parameters produces

$$\begin{aligned} \sigma^2 &= \frac{E[Z^4] - 3E^2[Z^2]}{6E[Z^2]} \\ \nu &= \frac{E[Z^2]}{2\sigma^2} - 1. \end{aligned}$$

The method of moments estimator for σ^2 and ν is obtained by replacing the second- and fourth-order moments by the sample moments.

For the simulation we used $\nu = 2$, $\sigma^2 = 3$, $N = 1000$ data samples, and $\omega_k = 0.01k$ for $k = 0, 1, \ldots, 500$. We searched over $1 \leq \sigma^2 \leq 5$ in maximizing (**??**). To assess the variance performance we also computed the Cramer-Rao lower bound (CRLB). From [3] it was found that for $\nu = 2$

$$\begin{aligned} CRLB(\hat{\nu}) &= \frac{620}{N} \\ CRLB(\sigma^2) &= \frac{80(\sigma^2)^2}{N}. \end{aligned}$$

The results are shown in Table 1. Note that the CGF estimator has less bias and variance for both

Table 1: Estimator performance for $\nu = 2$, $\sigma^2 = 3$, and $N = 1000$

	ν - mean	ν - variance	σ^2 - mean	σ^2 - variance
Moments estimator	2.6857	3.7867	2.9198	1.5183
CGF estimator	2.3595	1.5188	2.9974	0.9557
CRLB	− − −	0.6200	− − −	0.7200

parameters. Its variance performance versus the CRLB is an increased factor of $1.5188/0.6200 = 2.45$ for ν and $0.9557/0.7200 = 1.32$ for σ^2 while that for the method of moments estimator is substantially higher.

References

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3. Kay, S., C. Xu, "Cramer-Rao Lower Bound Computation via the Characteristic Function with Application to the K-Distribution", to be published in *IEEE Trans. on Aerospace and Electronics*.