Optimal Detector and Signal Design for STAP based on the Frequency-Wavenumber Spectrum

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Abstract

In this paper we address the design of an optimal transmit signal and its corresponding optimal detector for a radar system. The focus is on the spatial-temporal aspects of the waveform and is accomplished using the concept of the frequency-wavenumber spectrum. A uniform line array is assumed but the results are easily extended to a uniform planar array. The target is assumed to be a zero Doppler target with known range and bearing extent and the clutter is a stationary Gaussian random process with a known frequency-wavenumber spectrum. The advantage of the proposed approach is that a simple analytical result is obtained which is guaranteed to be optimal. Also, the current computational complexity of STAP, which requires the inversion of a large dimension matrix, can be reduced by using FFTs.

1 Introduction

We extend the results of our previous work that addressed optimal detector and transmit signal design for a temporal signal [1]. We now present the optimal detector and signal design for a space-time signal transmitted and received using a linear array. The results are directly applicable to STAP [2].

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The problem of signal waveform design for optimal detection in signal-dependent noise has been a problem of long-standing interest. In particular, the fields of radar and sonar have seen much work in this area. Some of the salient references are listed in [3]–[11]. Signal-dependent noise is generally referred to as clutter in radar and reverberation in active sonar. In either case, the fact that the received noise characteristics are dependent on the transmitted signal greatly complicates the signal design. For the case of signal design in colored noise whose spectrum *does not* depend on the transmitted signal, the solution is well known. It says to place all the signal energy into the frequency band for which the noise power is minimum. Correspondingly, for a discrete signal vector one should choose the signal as the eigenvector of the noise covariance matrix whose eigenvalue is minimum [13].

Few, if any, papers have addressed signal design for the general problem of spatial-temporal radar detection for signals in signal-dependent noise. In this paper we describe an approach that yields a simple solution, subject to the assumptions of a particular scattering model. The scattering model assumes that the signal-dependent noise is the output of a random linear time/spatially invariant (LTSI) spatial-temporal filter, whose impulse response can be assumed to be a realization of a two-dimensional (2-D) wide sense stationary (WSS) Gaussian random process. It should be noted that this model does not allow for spectral spreading, as would be inherent in an intrinsic clutter motion situation. It does, however, allow us to address the clutter characteristics associated with a moving platform and hence can be used for many STAP applications.

For a practical implementation one can envision a probing signal that measures the channel characteristics needed for waveform design. Then, the optimal transmit signal may be designed "in-situ". Techniques such as reported in [14] for channel estimation then become immediately applicable.

The paper is composed of the following. Section 2 presents the problem statement and the modeling assumptions employed. The optimal detector and its performance for a given transmit signal is described in Section 3. The signal that maximizes the detection performance is stated in Section 4. The application to STAP is the subject of Section 5 while Section 6 gives an explicit signal design example to combat clutter. Finally, a discussion and conclusions are given in Section 7.

2 Problem Statement and Modeling Assumptions

The model for the real bandpass received waveform is shown in Figure 1. A target that can have range and bearing extent is assumed so that $\bar{g}(t,x) = A\bar{q}(t,x)$, where $\bar{q}(t,x)$ is deterministic and known and A is a random reflection factor (the "overbar" will denote a real bandpass process with its absence denoting the complex envelope to be introduced later). We assume that the received waveform is a real bandpass process and is denoted by $\bar{z}(t,x)$ for $|t| \le T/2$ and $|x| \le L/2$. Initially,

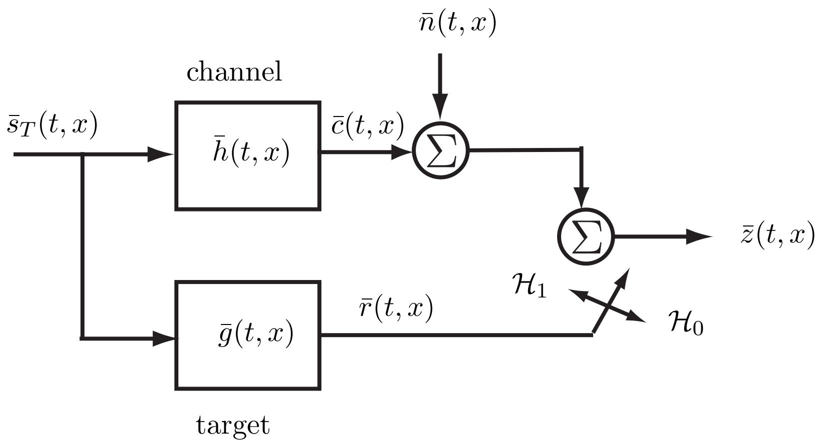

Figure 1: Modeling of real bandpass received waveform. $\bar{s}_T(t,x)$ is the transmitted signal, $\bar{h}(t,x)$ is the impulse response of the random LTSI channel filter, $\bar{g}(t,x)$ is the impulse response of the random LTSI target filter, and $\bar{n}(t,x)$ represents ambient noise and interference.

a receive linear aperture in space is assumed and hence the use of x, but is easily replaced by a uniform linear array (ULA), i.e., a sampled version in space. When no target return is present, i.e., under hypothesis $\mathcal{H}_0$, we have that $\bar{z}(t,x) = \bar{c}(t,x) + \bar{n}(t,x)$, where $\bar{c}(t,x)$ denotes clutter (henceforth, we will use radar terminology) and $\bar{n}(t,x)$ is the sum of ambient noise and interference, i.e., jamming. Under the hypothesis $\mathcal{H}_1$, the target return is modeled using $\bar{g}(t,x) = A\bar{q}(t,x)$ so that $\bar{r}(t,x) = A\bar{s}_T(t,x) * *\bar{q}(t,x)$, where "$**$" denotes two-dimensional (2-D) convolution. We have assumed a *zero Doppler target*. This is because it is not possible to design a transmit signal for each possible Doppler shift due to target motion. As the most difficult scenario is that of a zero Doppler target, we have chosen to address this problem. It is felt that if we can make progress on this signal design problem, then the nonzero Doppler target should yield improved performance as well. Note that it is only the optimality of the transmit signal that is in question for nonzero Doppler targets. The proposed detector is still applicable to the nonzero Doppler target but of course will require separate Doppler channels. Also, $\bar{n}(t,x)$ is modeled as a 2-D WSS Gaussian random process with zero mean and frequency-wavenumber power spectral density (FWPSD) $P_{\bar{n}}(F,K)$ [12]. Here F denotes the temporal frequency in Hz and K denotes wavenumber component in cycles/m. The positive frequency band is assumed to be $F_0 - W/2 \leq F \leq F_0 + W/2$, where F_0 is the transmit signal center frequency and hence all FWPSDs are defined over this band. Finally, we model the clutter return $\bar{c}(t,x)$ as the output of a *random LTSI spatial-temporal filter* with impulse response $\bar{h}(t,x)$, and whose input is the transmitted signal. Its FWPSD is given as $P_{\bar{h}}(F,K)$. Note that Doppler spreading due to intrinsic clutter motion is not accommodated, although platform motion can be incorporated. To model the former the more usual model is a *convolution in frequency*, which yields frequency spreading, as opposed to a multiplication. A further extension that will model intrinsic clutter motion is the use of a random *time varying* and spatially invariant filter. We do not pursue this further. Initially, we will also assume that the platform is at rest and latter extend the results to allow for platform motion inherent in airborne MTI radars.

Continuing with the clutter modeling, if $\bar{s}_T(t,x)$ is the transmit signal, then the clutter return will be $\bar{c}(t,x) = \bar{s}_T(t,x) \star\star \bar{h}(t,x)$ at the receiver. By reversing the convolution we can write this as $\bar{c}(t,x) = \bar{h}(t,x) \star\star \bar{s}_T(t,x)$, where now the filter input is $\bar{h}(t,x)$ and the filter impulse response is $\bar{s}_T(t,x)$. If we now assume that $\bar{h}(t,x)$ is a 2-D WSS Gaussian random process with zero mean and FWPSD $P_{\bar{h}}(F,K)$, then $\bar{c}(t,x)$ will also be a 2-D WSS Gaussian random process [16] with zero mean and FWPSD $P_{\bar{c}}(F,K) = LT|\bar{S}_T(F,K)|^2 P_{\bar{h}}(F,K)$, where $\bar{S}_T(F,K)$ is the normalized 2-D Fourier transform (the usual Fourier transform multiplied by $1/\sqrt{LT}$) of $\bar{s}_T(t,x)$ – see (1)).

3 Optimal Detector and its Performance for a Given Transmit Signal

Previously, all signals and noises have been defined as real bandpass processes. Their support in the F-K domain is shown in Figure 2 [12]. This represents the spectrum obtained by sensing a 3-D space-time field with a linear aperture. Hence, K is the wavenumber component along the aperture direction, which is assumed to be along the x axis. With the previous modeling assumptions and

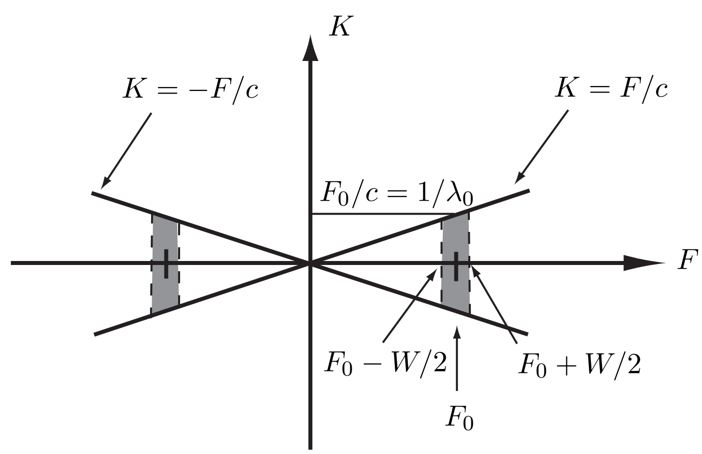

Figure 2: Support of real bandpass FWPSD around transmit frequency shown as the shaded region.

assuming that the time-bandwidth product WT satisfies $WT > 16$ and the spatial equivalent of the "time-bandwidth product" $(2/\lambda_0)L > 16$, where λ_0 is the wavelength at the center frequency F_0 (a narrowband assumption is always made), we can easily derive an optimal detector.

We first define the FW spectrum or 2-D Fourier transform of the received real bandpass process as

$$\bar{Z}(F,K) = \frac{1}{\sqrt{LT}} \int_{-L/2}^{L/2} \int_{-T/2}^{T/2} \bar{z}(t,x) \exp(-j2\pi(Ft - Kx)) dt dx \tag{1}$$

where the negative frequencies are redundant since $\bar{Z}(-F,-K) = \bar{Z}^*(F,K)$ for real $\bar{z}(t,x)$. Considering only the positive frequency components, we can rewrite the Fourier transform of the real bandpass process as $\bar{Z}(F+F_0,K)$, where now $|F| \leq W/2$, which are the baseband frequencies. Then, we have

$$\begin{aligned} \bar{Z}(F+F_0,K) &= \frac{1}{\sqrt{LT}} \int_{-L/2}^{L/2} \int_{-T/2}^{T/2} \bar{z}(t,x) \exp[-j2\pi((F+F_0)t - Kx)] dt dx \\ &= \frac{1}{\sqrt{LT}} \int_{-L/2}^{L/2} \int_{-T/2}^{T/2} \underbrace{[\bar{z}(t,x)\exp(-j2\pi F_0 t)]_{LPF}}_{z(t,x)} \exp(-j2\pi(Ft - Kx)) dt dx. \end{aligned}$$

Since we are only concerned with the positive frequency components, the lowpass filtering, denoted above by "LPF", will not affect the transform. Now $z(t,x)$ is the complex envelope of $\bar{z}(t,x)$ [15] and thus we have that $\bar{Z}(F+F_0,K) = Z(F,K)$ for $|F| \leq W/2$, where $Z(F,K)$ is just the Fourier transform of the complex envelope. The support for the FW spectrum of the complex envelope is shown in Figure 3. As a result the Neyman-Pearson detector, which is derived in Appendix A, is

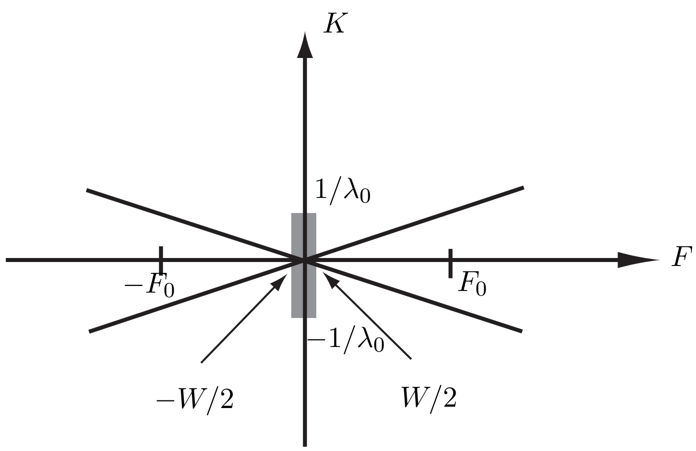

Figure 3: Support of FW spectrum of complex envelope $z(t,x)$ (which is obtained from $\bar{z}(t,x)$ by demodulation to baseband and lowpass filtering). The received bandpass signal $\bar{z}(t,x)$ is assumed to be narrowband.

given in term of the baseband frequencies as

$$\left| \sum_{m=-M/2}^{M/2} \sum_{n=-N/2}^{N/2} \frac{Z(F_m,K_n) S_T^*(F_m,K_n) \sqrt{LT} Q_t^*(F_m,K_n)}{P_h(F_m,K_n) LT |S_T(F_m,K_n)|^2 + P_n(F_m,K_n)} \right|^2 > \gamma \tag{2}$$

where $F_m = m/T$ with $M = WT$, and $K_n = n/L$ with $N = (2/\lambda_0)L$ and the baseband FWPSDs are $P_h(F,K) = [P_{\bar{h}}(F+F_0,K)]_{LPF}$, $P_n(F,K) = [P_{\bar{n}}(F+F_0,K)]_{LPF}$, $Q_t(F,K)$ is the FW Fourier transform of $q(t,x)$ (with the subscript "t" denoting the target, not time), which is the complex envelope of $\bar{q}(t,x)$, and $S_T(F,K)$ is the FW Fourier transform of the transmitted signal complex envelope. It is seen that the detector consists of a prewhitener in frequency and wavenumber, followed by a matched filter to the signal $\sqrt{LT}S_T(F,K)Q_t(F,K)$, and a magnitude-squaring operation. In practice, FFTs would be used to approximate the continuous-time Fourier transforms used in (2). The detection performance of the optimal detector is shown in Appendix A to be monotonically increasing with the deflection coefficient

$$d^2 = \sigma_A^2 \int_{-1/\lambda_0}^{1/\lambda_0} \int_{-W/2}^{W/2} \frac{LT|S_T(F,K)|^2 LT|Q_t(F,K)|^2}{P_h(F,K)LT|S_T(F,K)|^2 + P_n(F,K)} dFdK. \tag{3}$$

Hence, the optimal signal design problem reduces to the relatively simple problem of choosing a transmit signal $s_T(t,x)$ that is constrained in energy, which is defined as

$$\mathcal{E} = \int_{-1/\lambda_0}^{1/\lambda_0} \int_{-W/2}^{W/2} LT|S_T(F,K)|^2 dFdK \tag{4}$$

and that maximizes d^2. Note that it is only the energy spectral density (ESD) or $\mathcal{E}_s(F,K) = LT|S_T(F,K)|^2$ that affects performance. The phase of the Fourier transform can be chosen arbitrarily and in practice will be selected for ease of signal realizability.

4 Maximizing the Detection Performance by Transmit Signal Design

The key to maximizing (3) over all $|S_T(F,K)|^2$, subject to the constraints that $|S_T(F,K)|^2 \geq 0$ and the energy constraint of (4), lies in the property that d^2 is a *concave functional* of $|S_T(F,K)|^2$. This assures us that the solution found via differential means will produce a *global maximum*. However, with the proposed approach, we will only find $|S_T(F,K)|^2$, so that a further necessary step is to synthesize a space/time-limited signal with the given ESD. Fortunately, this is possible and amounts to a filter design problem based on a given magnitude frequency response specification (see also Section 6.3 for more details for STAP). Many techniques are available to effect the one-dimensional designs [17], with 2-D designs as extensions. Future work will address the signal design problem.

The ESD that maximizes d^2 is found in a similar fashion to that in [1] as

$$\mathcal{E}_s(F,K) = LT|S_T(F,K)|^2 = \max\left(\frac{\sqrt{LT|Q_t(F,K)|^2 P_n(F,K)/\alpha} - P_n(F,K)}{P_h(F,K)}, 0\right) \tag{5}$$

where $\max(x,0)$ means the maximum of x and 0. The parameter α is found from the energy constraint of (4) so that we must solve

$$\int_{-1/\lambda_0}^{1/\lambda_0} \int_{-W/2}^{W/2} \max\left(\frac{\sqrt{LT|Q_t(F,K)|^2 P_n(F,K)/\alpha} - P_n(F,K)}{P_h(F,K)}, 0\right) dFdK = \mathcal{E} \tag{6}$$

for α, where α is positive. A solution for α is guaranteed since if

$$g(\alpha) = \int_{-1/\lambda_0}^{1/\lambda_0} \int_{-W/2}^{W/2} \max\left(\frac{\sqrt{LT|Q_t(F,K)|^2 P_n(F,K)/\alpha} - P_n(F,K)}{P_h(F,K)}, 0 \right) dF dK$$

then $g(0) = \infty$ and $g(\infty) = 0$ and g is a continuous function, which means that it takes on all values in between by the intermediate value theorem. We can narrow down the search region for α, however, by noting that for $\mathcal{E} > 0$, we must have $\frac{\sqrt{LT|Q_t(F,K)|^2 P_n(F,K)/\alpha} - P_n(F,K)}{P_h(F,K)} > 0$ for at least some values of (F, K). We can then exclude those values of α for which

$$\frac{\sqrt{LT|Q_t(F,K)|^2 P_n(F,K)/\alpha} - P_n(F,K)}{P_h(F,K)} \leq 0$$

for all (F, K). These are the values $\alpha \geq LT|Q_t(F,K)|^2/P_n(F,K)$. Thus, we need not search the values for which $\alpha \geq \max LT|Q_t(F,K)|^2/P_n(F,K)$ or we have that the search region is

$$0 < \alpha < \max \frac{LT|Q_t(F,K)|^2}{P_n(F,K)}.$$

We can also compute the maximum value of d^2 to allow us to determine improvements over other detectors, either in the case of a suboptimal detector, e.g., a matched filter, or in the case of the optimal detector that uses a suboptimal transmit signal. The maximum value of d^2 is given by (3) with the optimal signal ESD given by (5).

5 STAP Modeling

5.1 Target Return Modeling

It is assumed that the platform is moving at a constant velocity v in the x-direction and that the transmit/receive array is a uniformly spaced line array (ULA). The array is also aligned along the x-direction. For simplicity we assume a 2-D geometry with the more general case found by a simple extension [2]. As a result, the geometry is illustrated in Figure 4. The ensonified portion of the environment at range R is shown as the solid semicircle. For the initial development we assume a single omnidirectional transmitter at location x_T and a stationary platform ($v = 0$). It transmits a real bandpass signal $\bar{s}_T(t)$ for $|t| \leq T/2$, whose frequency band is given by $B = \{F : |F \pm F_0| \leq W/2\}$. As before, we will assume a narrowband signal. A target located at a range R and bearing θ from the array center reflects the signal and at location x an omnidirectional receiver produces a received signal denoted by $\bar{r}(t, x)$. Assuming a target in the far field, the round trip delay to a receiver located at x is $\tau(x) = 2R/c - (x_T + x)\cos(\theta)/c$, where c is the speed of propagation. As a result, the received bandpass signal is $\bar{r}(t, x) = \bar{s}_T(t - \tau(x))$. For the time being we have assumed a perfect reflector. Later we will allow the target to exhibit a finite spatial and temporal extent.

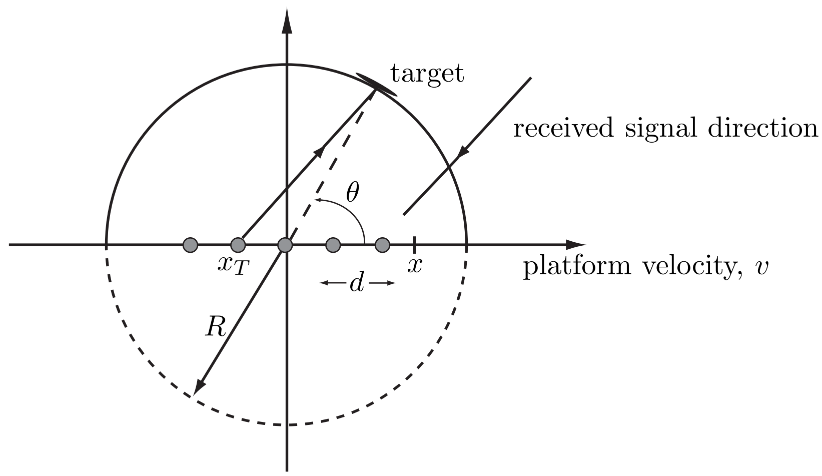

Figure 4: Geometry of line array.

No account is taken of the propagation loss and/or dispersion. Thus we have

$$\begin{aligned}\bar{r}(t,x) &= \sqrt{T}\int_B \bar{S}_T(F)\exp[-j2\pi F\tau(x)]\exp(j2\pi Ft)dF \\ &= \sqrt{T}\int_B \bar{S}_T(F)\exp[-j2\pi(F/c)2R]\exp[j2\pi(F/c)(x+x_T)\cos\theta]\exp(j2\pi Ft)dF.\end{aligned}$$

Next assume that the bulk delay, i.e., $\tau(0) = 2R/c$, is accounted for in the processor so that we have

$$\bar{r}(t,x) = \sqrt{T}\int_B \bar{S}_T(F)\exp[j2\pi(F/c)(x+x_T)\cos\theta]\exp(j2\pi Ft)dF.$$

For a propagating monochromatic plane wave the signal is represented as $\exp[j2\pi(Ft - \mathbf{K}^T\mathbf{r})]$, where $\mathbf{K}$ is the vector wavenumber and points in the direction of the transmitted signal toward the target, and $\mathbf{r}$ is the position vector of measurement. The x-component of $\mathbf{K}$ will be denoted by $K^{(o)}$ and represents the wavenumber component of the *outgoing wave*. Note that $K^{(o)}$ is given by $K^{(o)} = (F/c)\cos\theta$ so that for $0 \le \theta \le \pi$, we have that $-F/c \le K^{(o)} \le F/c$. Using this we have

$$\bar{r}(t,x) = \sqrt{T}\int_B \bar{S}_T(F)\exp[j2\pi K^{(o)}(x+x_T)]\exp(j2\pi Ft)dF.$$

For a more general target response the reflectivity will be a function of frequency and direction of incident arrival angle. To incorporate this more realistic model we assume that an incident unit amplitude complex monochromatic plane wave at frequency F_0 and x-component of wavenumber $K_0^{(o)}$ is backscattered from the target and toward the receiver with an amplitude $A\bar{Q}_t(F_0, -K_0^{(o)})$ (the "t" subscript refers to the target and not time), where as before $A \sim \mathcal{CN}(0, \sigma_A^2)$. Hence, an incident plane wave $\exp[j2\pi(F_0 t - K_0^{(o)}x)]$ is backscattered in the reverse direction yielding $A\bar{Q}_t(F_0, -K_0^{(o)})\exp[j2\pi(F_0 t + K_0^{(o)}x)]$. In order to reference the arrival angle as seen by the receive array as incoming, we let $K_0 = -K_0^{(o)}$, which now represents the x-component of an *arriving* plane wave and is $K_0 = -(F_0/c)\cos\theta$. With these modifications we have that

$$\bar{r}(t,x) = A\sqrt{T}\int_B \bar{S}_T(F)\bar{Q}_t(F,K_0)\exp[-j2\pi K_0(x+x_T)]\exp(j2\pi Ft)dF.$$

Considering next a *finite length* target so that the backscatter is over a sector of receiving azimuthal angles $0 \le \theta \le \pi$, i.e., over K, and for a receive aperture of L m, we have the sum of the reflections being expressable as

$$\bar{r}(t,x) = A\sqrt{LT}\int_B\int_{-F/c}^{F/c} \bar{S}_T(F)\bar{Q}_t(F,K)\exp(-j2\pi Kx_T)\exp(j2\pi Ft)\exp(-j2\pi Kx)dKdF.$$

Next we assume a discrete set of transmitters, i.e., an ULA, with transmitters at positions x_{T_n} for $n = -N/2,\ldots,N/2$. The transmit signal at the nth transmitter in the frequency domain is denoted by $\bar{S}_{T_n}(F)$. Note that we have allowed for a different signal to be transmitted from each transmitter. The received signal becomes by superposition

$$\bar{r}(t,x) = A\sqrt{LT}\int_B\int_{-F/c}^{F/c}\sum_{n=-N/2}^{N/2} \bar{S}_{T_n}(F)\bar{Q}_t(F,K)\exp(-j2\pi Kx_{T_n})\exp[j2\pi(Ft-Kx)]dKdF$$

and it is now seen that the frequency-wavenumber (FW) spectrum of the received real bandpass signal is

$$A\sum_{n=-N/2}^{N/2}\bar{S}_{T_n}(F)\bar{Q}_t(F,K)\exp(-j2\pi Kx_{T_n}) \quad |F\pm F_0| \le W/2;\ -F/c \le K \le F/c.$$

Upon demodulating the received signal to baseband and lowpass filtering, we obtain the complex envelope signal as

$$\begin{aligned} r(t,x) &= [\bar{r}(t,x)\exp(-j2\pi F_0 t)]_{LPF} \\ &= \Bigg[A\sqrt{LT}\int_{F_0-W/2}^{F_0+W/2}\int_{-F/c}^{F/c}\sum_{n=-N/2}^{N/2}\bar{S}_{T_n}(F)\bar{Q}_t(F,K)\exp(-j2\pi Kx_{T_n}) \\ &\qquad \cdot\exp[j2\pi((F-F_0)t-Kx)]dKdF\Bigg]_{LPF} \end{aligned}$$

and letting $F' = F - F_0$, this becomes

$$\begin{aligned} r(t,x) &= \Bigg[A\sqrt{LT}\int_{-W/2}^{W/2}\int_{-(F'+F_0)/c}^{(F'+F_0)/c}\bar{Q}_t(F'+F_0,K)\sum_{n=-N/2}^{N/2}\bar{S}_{T_n}(F'+F_0) \\ &\qquad \cdot\exp(-j2\pi Kx_{T_n})\exp[j2\pi(F't-Kx)]dKdF'\Bigg]_{LPF} \end{aligned}$$

and letting $Q_t(F,K) = \bar{Q}_t(F+F_0,K)$ and $S_{T_n}(F) = \bar{S}_{T_n}(F+F_0)$, and invoking the narrowband assumption, we have finally that

$$\begin{aligned} r(t,x) &= A\sqrt{LT}\int_{-W/2}^{W/2}\int_{-1/\lambda_0}^{1/\lambda_0} Q_t(F,K)\sum_{n=-N/2}^{N/2} S_{T_n}(F)\exp(-j2\pi K x_{T_n}) \\ &\quad \cdot \exp[j2\pi(Ft-Kx)]dKdF \end{aligned} \tag{7}$$

which says that at baseband the FW spectrum of the received complex envelope signal is

$$\begin{aligned} R(F,K) &= AQ_t(F,K)\sum_{n=-N/2}^{N/2} S_{T_n}(F)\exp(-j2\pi K x_{T_n}) \\ &= AQ_t(F,K)S_T(F,K) \end{aligned} \tag{8}$$

where $S_{T_n}(F)$ is the Fourier transform of the transmitted complex envelope signal from transmitter n, and $S_T(F,K)$ can be interpreted as the overall transmit signal.

5.2 Clutter Modeling

We can use the same modeling for clutter except that the returns will be more widely distributed in azimuth. Replacing the reflectivity factor, $AQ_t(F,K)$, of the target in (7) by $Q_h(F,K)$, we assume that the random space-time impulse response, which is $\mathcal{F}^{-1}\{Q_h(F,K)\}$, is a wide sense stationary complex Gaussian random process. As a result it can be shown that $Q_h(F,K) \sim \mathcal{CN}(0, P_h(F,K))$, where $\mathcal{CN}$ denotes a complex Gaussian probability density function and $P_h(F,K)$ is a FW power spectral density (PSD). It follows that the FW PSD of the clutter, which will be denoted by $P_c(F,K)$, can be written as

$$\begin{aligned} P_c(F,K) &= LT\left|\sum_{n=-N/2}^{N/2} S_{T_n}(F)\exp(-j2\pi K x_{T_n})\right|^2 P_h(F,K) \qquad |F| \le W/2; |K| \le \frac{1}{\lambda_0} \\ &= LT|S_T(F,K)|^2 P_h(F,K). \end{aligned} \tag{9}$$

We must be careful to note that in specifying $P_h(F,K)$, the frequency variable corresponds to a physical scattering mechanism at $F = F_0$ and not at baseband. In the next section we present some examples to verify the reasonableness of the derived models.

6 Some Examples

6.1 Target Modeling

Consider that we transmit the same signal from all transmitters and steer a beam in the direction of θ_0. To do so the real bandpass signal transmitted from the nth transmitter is given as $\bar{s}_{T_n}(t) =$

$\bar{s}_T(t-\tau_n)$, where $\tau_n = (x_{T_n}/c)\cos(\theta_0)$. As a result, the transmitted signal is given in the Fourier domain as

$$\bar{S}_{T_n}(F) = \bar{S}_T(F)\exp[-j2\pi F(x_{T_n}/c)\cos(\theta_0)] = \bar{S}_T(F)\exp[-j2\pi K_0^{(o)} x_{T_n}]$$

where $K_0^{(o)} = (F/c)\cos(\theta_0)$ is the outgoing wavenumber. Converting to an incoming wavenumber produces $K_0 = -K_0^{(o)}$ and converting to the complex envelope transmit signal

$$S_{T_n}(F) = S_T(F)\exp(j2\pi K_0 x_{T_n})$$

where $K_0 = -(F/c)\cos(\theta_0)$. Using (7) we have

$$\begin{aligned} r(t,x) &= A\sqrt{LT}\int_{-W/2}^{W/2}\int_{-1/\lambda_0}^{1/\lambda_0} Q_t(F,K)S_T(F)\sum_{n=-N/2}^{N/2}\exp(-j2\pi(K-K_0)x_{T_n}) \\ &\quad \cdot\exp[j2\pi(Ft-Kx)]dKdF \end{aligned} \tag{10}$$

and thus

$$R(F,K) = AQ_t(F,K)S_T(F)\sum_{n=-N/2}^{N/2}\exp(-j2\pi(K-K_0)x_{T_n}). \tag{11}$$

By using the same transmit signal at each transmitter the overall transmitted signal is seen to be separable in space and time. A more general signal is needed to realize an arbitrary space-time transmit signal as discussed in Section 5.3. Also, the received FW of the signal is maximum in amplitude when $K = K_0$ as expected.

6.2 Clutter Modeling

In general, the FWPSD for the clutter will be from (9)

$$P_c(F,K) = LT|S_T(F,K)|^2 P_h(F,K)$$

and corresponds to the PSD of the clutter space-time random process $c(t,x)$ observed along the line array, where $S_T(F,K)$ is given by (9). As an example, for narrowband transmit signals, then $P_h(F,K)$ will probably not depend on the frequency deviation from $F = F_0$. As a result, for isotropic scattering we would have $P_h(F,K) = P_0$ and thus,

$$P_c(F,K) = P_0 LT|S_T(F,K)|^2.$$

Alternatively, for the often used model of clutter patches (a finite set of discrete scatterers) with each one having a power of P_i, we would have

$$P_h(F,K) = \sum_{i=1}^{I}\frac{P_i}{\sqrt{L}}\delta(K-K_i).$$

This will produce a clutter FWPSD of

$$P_c(F,K) = \sum_{i=1}^{I}\frac{P_i}{\sqrt{L}}LT|S_T(F,K_i)|^2\delta(K-K_i).$$

6.3 Signal Generation for ULA

We assume a ULA with half wavelength spacing at $F = F_0$ of $\Delta_x = \lambda_0/2$ and also a pulse train with pulse repetition interval of $\Delta_t = 1/W$. Then, we have that

$$S_{T_n}(F) = \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} s_{T_n}(t) \exp(-j2\pi F t) dt \qquad |F| \leq W/2$$

and the pulse train can be represented as

$$s_{T_n}(t) = \sqrt{T} \sum_{p=-P/2}^{P/2} s_n[p] \delta(t - p\Delta_t) \quad n = -N/2, \ldots, N/2.$$

Thus,

$$S_{T_n}(F) = \sum_{p=-P/2}^{P/2} s_n[p] \exp(-j2\pi F p \Delta_t)$$

and we have that

$$\begin{aligned} S_T(F, K) &= \sum_{n=-N/2}^{N/2} S_{T_n}(F) \exp(-j2\pi K x_{T_n}) \\ &= \sum_{n=-N/2}^{N/2} \sum_{p=-P/2}^{P/2} s_n[p] \exp(-j2\pi F p \Delta_t) \exp(-j2\pi K n \Delta_x) \\ &= \sum_{p=-P/2}^{P/2} \sum_{n=-N/2}^{N/2} s_n[p] \exp[-j2\pi(F p \Delta_t + K n \Delta_x)]. \end{aligned}$$

If we normalize the frequencies to $f = F\Delta_t$ and $k = K\Delta_x$, then we have finally that $|f| \leq 1/2$ and $|k| \leq 1/2$. This yields

$$S_T^{(D)}(f, k) = \sum_{p=-P/2}^{P/2} \sum_{n=-N/2}^{N/2} s_n[p] \exp[-j2\pi(fp + kn)] \qquad -1/2 \leq f \leq 1/2;\ -1/2 \leq k \leq 1/2$$

which is the discrete-time Fourier transform of the two-dimensional sequence $s_n[p]$. Standard digital signal processing approaches can be used to synthesize this discrete space-time signal to yield a given energy spectral density.

7 Extension to a Moving Platform

Now assume that the platform is moving in the x-direction relative to the ground with speed v. The received complex envelope signal observed with respect to a ground reference becomes $r(t, x_p + 2vt)$, where x_p is the x-position relative to the platform array origin as shown in Figure 4. As a result, we have from (7)

$$\begin{aligned} r(t, x_p + 2vt) &= A\sqrt{LT} \int_{-W/2}^{W/2} \int_{-1/\lambda_0}^{1/\lambda_0} Q_t(F,K) \sum_{n=-N/2}^{N/2} S_{T_n}(F) \exp(-j2\pi K x_{T_n}) \\ &\quad \cdot \exp[j2\pi(Ft - K(x_p + 2vt))] dK dF \end{aligned} \tag{12}$$

or letting $r_p(t,x) = r(t, x_p + 2vt)$ be the received signal relative to the platform and where now x denotes the position relative to the origin of the ULA, we have

$$\begin{aligned} r_p(t,x) &= A\sqrt{LT} \int_{-W/2}^{W/2} \int_{-1/\lambda_0}^{1/\lambda_0} Q_t(F,K) \sum_{n=-N/2}^{N/2} S_{T_n}(F) \exp(-j2\pi K x_{T_n}) \\ &\quad \cdot \exp[j2\pi((F - 2Kv)t - Kx)] dK dF. \end{aligned}$$

Next let $F' = F - 2Kv$ so that

$$\begin{aligned} r_p(t,x) &= A\sqrt{LT} \int_{-W/2}^{W/2} \int_{-1/\lambda_0}^{1/\lambda_0} Q_t(F' + 2Kv, K) \sum_{n=-N/2}^{N/2} S_{T_n}(F' + 2Kv) \exp(-j2\pi K x_{T_n}) \\ &\quad \cdot \exp[j2\pi(F't - Kx)] dK dF' \end{aligned}$$

where the limits have not changed since we assume that the increase in bandwidth due to platform motion (or Doppler) is accommodated by the bandwidth W. Note that now the FW spectrum of the received complex envelope signal is

$$\begin{aligned} R(F,K) &= AQ_t(F + 2Kv, K) \sum_{n=-N/2}^{N/2} S_{T_n}(F + 2Kv) \exp(-j2\pi K x_{T_n}) \\ &= AQ_t(F + 2Kv, K) S_T(F + 2Kv, K) \qquad |F| \leq W/2 \end{aligned}$$

and that the clutter FWPSD will similarly be

$$P_c(F,K) = LT|S_T(F + 2Kv, K)|^2 P_h(F + 2Kv, K).$$

Note that this is the same receive signal and clutter FWPSD as before except that F is replaced by $F + 2Kv$ due to platform motion. If for example, the clutter is uniform in wavenumber, i.e., returns at all azimuthal angles ($0 \leq \theta \leq \pi$) and is narrow in frequency, i.e., for a CW transmit, then for no platform motion ($v = 0$) $P_h(F,K)$ will be maximum along $F = 0$ for all K as shown in Figure 5, assuming ground clutter. As a result, with platform motion the *clutter ridge* should appear when $F + 2Kv = 0$ or at $K = -F/(2v)$ as shown in Figure 6. Finally, if we plot $u = \cos(\theta)$ along the horizontal axis and F along the vertical axis, then Figure 6 becomes Figure 7. This is the usual clutter ridge that is seen after processing actual data, which can be considered *an estimate* of the received clutter FWPSD. To see this we have that the ridge is given by

$$F = -2Kv \approx 2\frac{F_0}{c}\cos(\theta)v = 2\frac{F_0}{c}vu$$

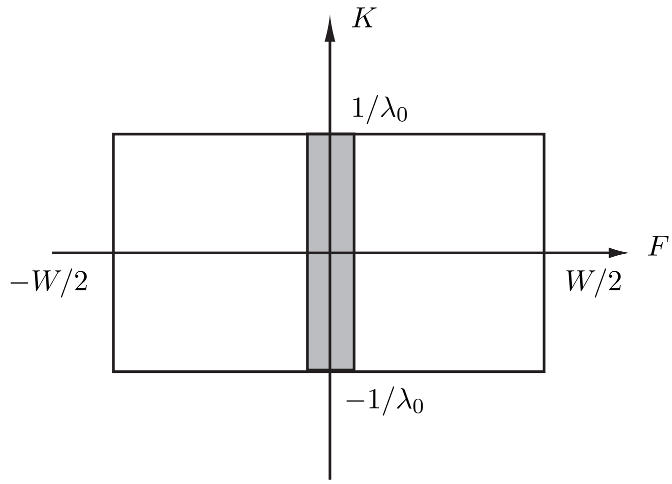

Figure 5: Frequency-wavenumber clutter ridge for no platform motion, i.e., support of $P_h(F, K)$.

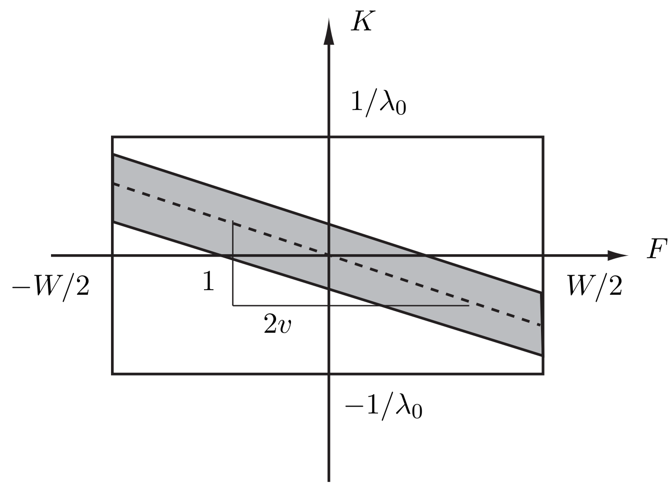

Figure 6: Frequency-wavenumber clutter ridge for platform motion, i.e., support of $P_h(F, K)$.

with the approximation used for a narrowband system. Thus, we have that

$$F = \beta u$$

where the slope in the frequency-angle (actually frequency-cosine-angle) domain is

$$\beta = \frac{2v}{c} F_0.$$

Clearly, for good detectability the target Doppler should cause the target to appear outside the clutter ridge.

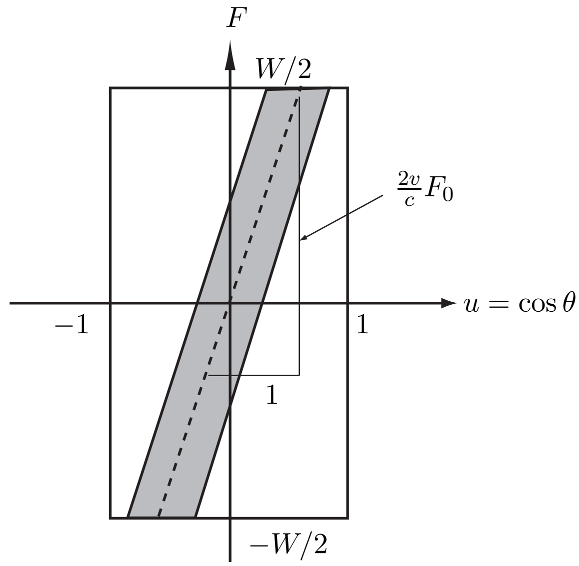

Figure 7: Frequency-angle clutter ridge for platform motion - usual depiction.

8 Detectability Improvement via Signal Design

In this section we describe an example of the improvement in detectability when the signal is chosen optimally to mitigate the effects of ground clutter. To do so we model the clutter as a sum of ground clutter and a discrete scatterer near the target. The platform is assumed to be stationary so that the ground clutter appears as in Figure 5. The system parameters chosen are a center frequency of $F_0 = 1$ Ghz, a signal bandwidth of $W = 1$ Mhz, an ambient noise background having a FWPSD of $P_n(F,K) = \sigma_n^2$, where $0.001 \leq \sigma_n^2 \leq 0.01$. The target response FWPSD is assumed to be $LT|Q_t(F,K)|^2 = 1$ for all $|F| \leq W/2$ and $|K| \leq 1/\lambda_0$. The FWPSD of the clutter is modeled by

$$\begin{aligned} P_h(F,K) &= \frac{P_g}{\sqrt{2\pi\sigma_g^2}} \exp\left[-\frac{1}{2\sigma_g^2}(F/W)^2)\right] \\ &+ \frac{P_d}{\sqrt{2\pi\sigma_d^2}} \exp\left[-\frac{1}{2\sigma_d^2}\left((F/W)^2 + ((K-K_d)/(1/\lambda_0))^2\right)\right] \qquad |F| \leq W/2; |K| \leq 1/\lambda_0 \end{aligned}$$

where the first term is the ground clutter and the second term is the discrete clutter return. Note that the ground clutter is uniform in wavenumber (doesn't depend on K) and has a spread in frequency determined by the value of σ_g. The discrete clutter component is circularly symmetric with an extent given by σ_d and centered at $(F,K) = (0, K_d)$. In the example to follow the parameters chosen were $P_g = 100, \sigma_g^2 = 0.1, P_d = 100, \sigma_d^2 = 0.001, K_d = 0.05(1/\lambda_0)$. The clutter FWPSD is shown in Figures 8 and 9. Because the ambient noise FWPSD $P_n(F,K)$ is constant as is the target FWPSD $Q_t(F,K)$, it is easily shown from (5) that the optimal signal is one for which

$$\mathcal{E}_{s_{\rm opt}}(F,K)P_h(F,K) = c \tag{13}$$

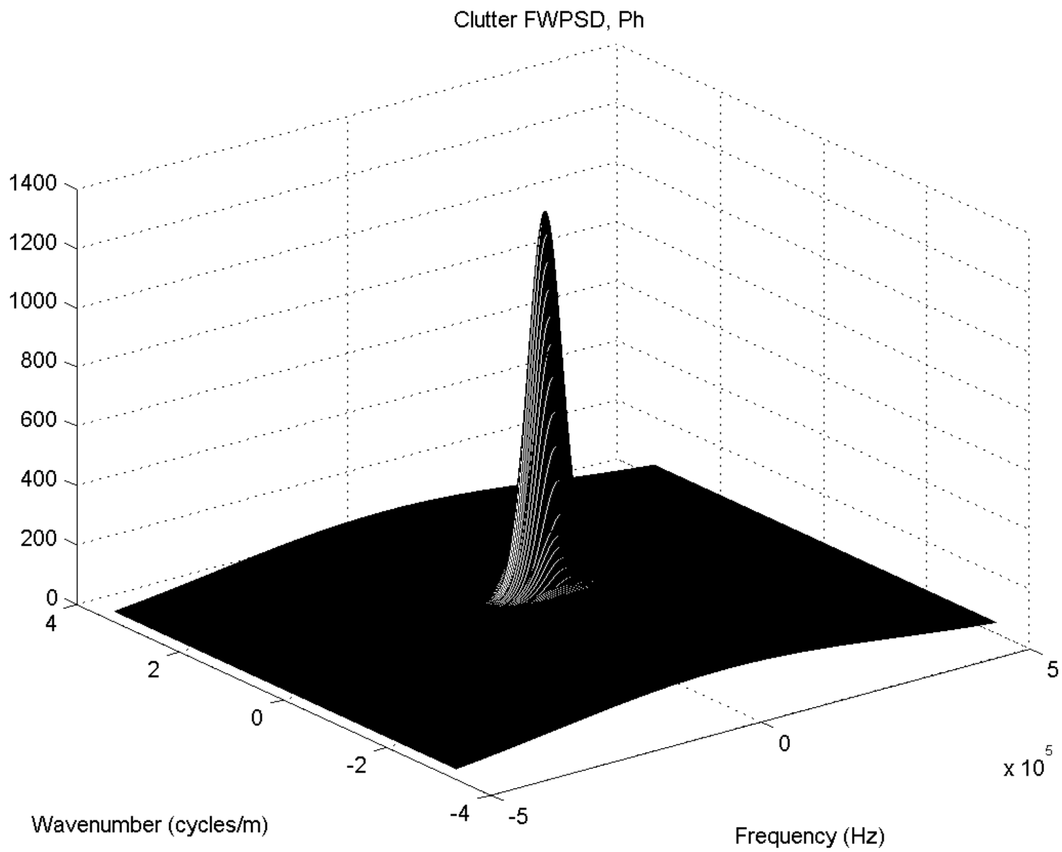

Figure 8: Frequency-wavenumber PSD for clutter, $P_h(F, K)$ - 3D plot.

where c is a constant. In effect, the signal is chosen to make the noise due to clutter *white in frequency and wavenumber*. The constant c then is specified to maintain the energy constraint. For this example the optimal transmit FW energy spectral density $\mathcal{E}_{s_{\text{opt}}}(F, K)$ is shown in Figure 10. It is seen to be inversely proportional to the clutter PSD by comparing Figure 10 to Figure 9.

As a comparison to a suboptimal signal we consider a signal with frequency-wavenumber energy spectral density (FWESD) given by

$$\mathcal{E}_s(F, K) = \frac{P_d}{\sqrt{2\pi\sigma_d^2}} \exp\left[-\frac{1}{2\sigma_d^2}\left((F/W)^2 + (K/(1/\lambda_0))^2\right)\right]$$

which has the same PSD as the discrete clutter component except that it is centered at $K = 0$ as opposed to $K = K_d$. As might be expected the lack of prewhitening causes the signal return energy to have to compete with the discrete clutter and hence the detection performance will be poor. To compare the performance we can use the deflection coefficients which for the optimal signal is given

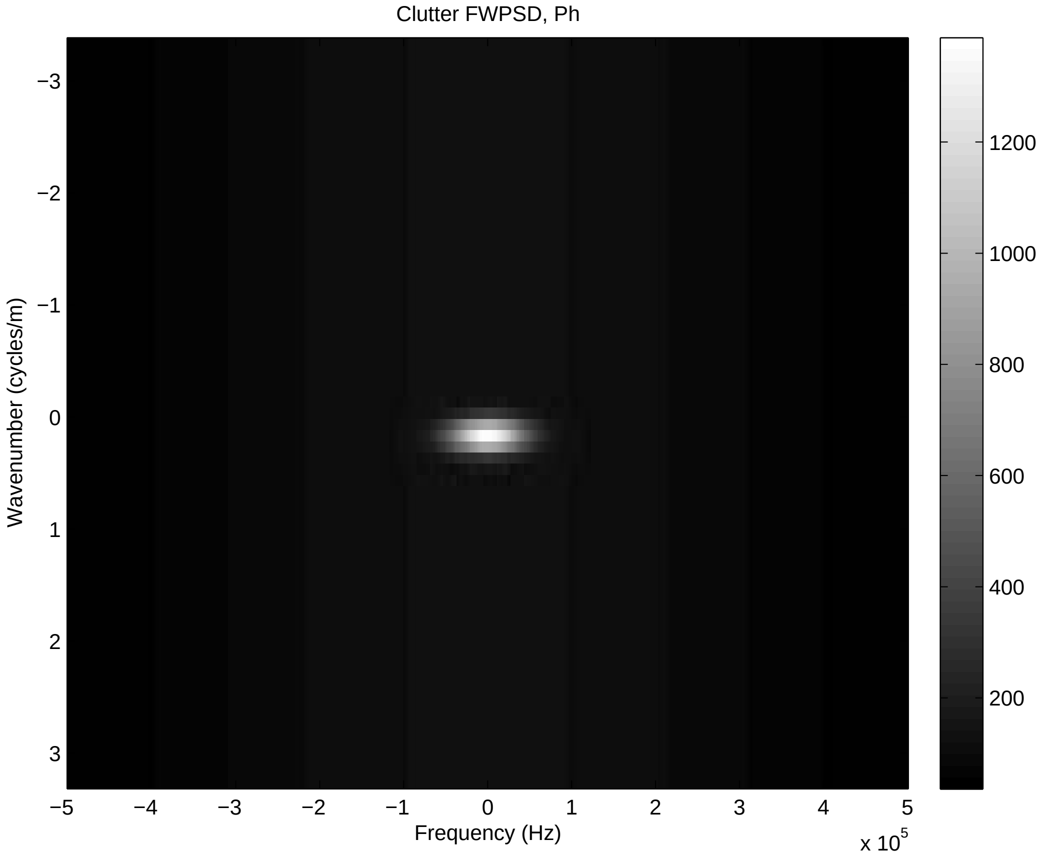

Figure 9: Frequency-wavenumber PSD for clutter, $P_h(F, K)$ - image plot.

from (3) and (13), and noting that $\mathcal{E}_s(F,K) = LT|S_T(F,K)|^2$ as

$$\begin{aligned} d^2_{\text{opt}} &= \sigma_A^2 \int_{-1/\lambda_0}^{1/\lambda_0} \int_{-W/2}^{W/2} \frac{LT|S_T(F,K)|^2 LT|Q_t(F,K)|^2}{P_h(F,K)LT|S_T(F,K)|^2 + P_n(F,K)} dF dK \\ &= \sigma_A^2 \int_{-1/\lambda_0}^{1/\lambda_0} \int_{-W/2}^{W/2} \frac{c/P_h(F,K)}{c + \sigma_n^2} dF dK. \end{aligned} \tag{14}$$

When the energy constraint is enforced we can solve for c using

$$\begin{aligned} \mathcal{E} &= \int_{-1/\lambda_0}^{1/\lambda_0} \int_{-W/2}^{W/2} \mathcal{E}_{s_{\text{opt}}}(F,K) dF dK \\ &= \int_{-1/\lambda_0}^{1/\lambda_0} \int_{-W/2}^{W/2} \frac{c}{P_h(F,K)} dF dK \end{aligned}$$

and therefore

$$c = \frac{\mathcal{E}}{\int_{-1/\lambda_0}^{1/\lambda_0} \int_{-W/2}^{W/2} \frac{1}{P_h(F,K)} dF dK}.$$

Thus, (14) becomes

$$d^2_{\text{opt}} = \sigma_A^2 \frac{\mathcal{E} \int_{-1/\lambda_0}^{1/\lambda_0} \int_{-W/2}^{W/2} 1/P_h(F,K) dF dK}{\mathcal{E} + \sigma_n^2 \int_{-1/\lambda_0}^{1/\lambda_0} \int_{-W/2}^{W/2} 1/P_h(F,K) dF dK} \tag{15}$$

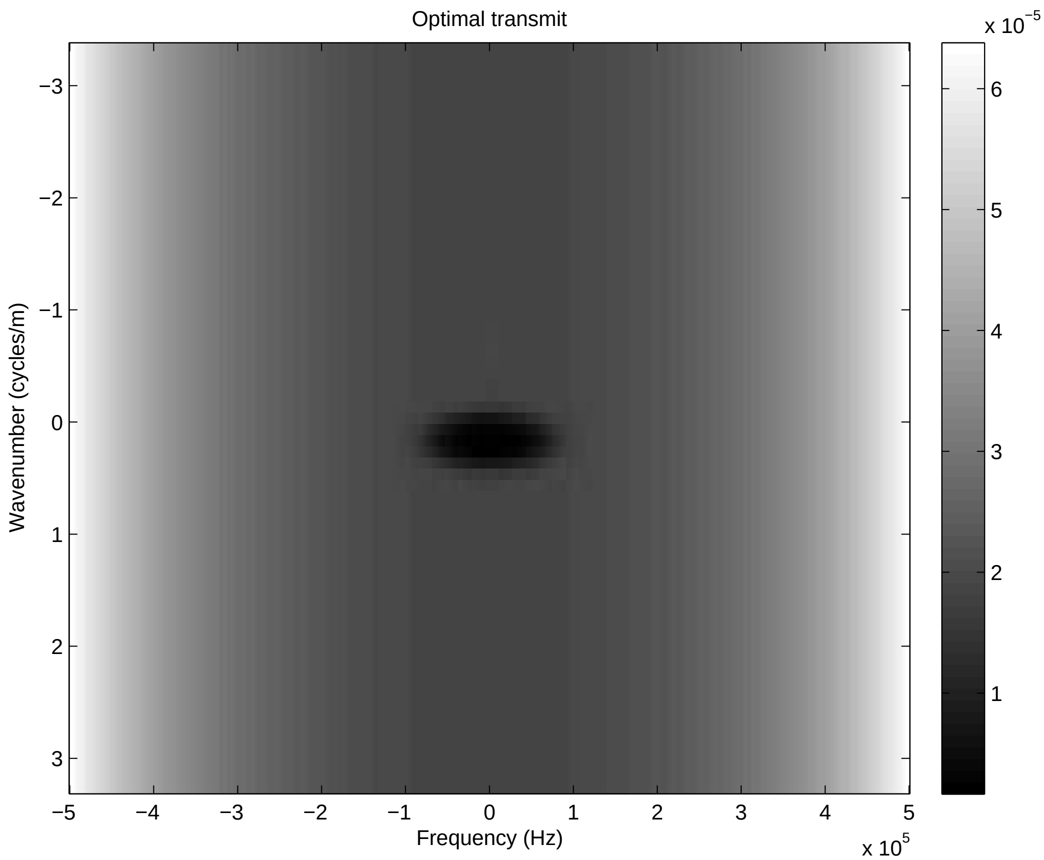

Figure 10: Frequency-wavenumber ESD $\mathcal{E}_{s_{\text{opt}}}(F,K)$ for optimal transmit signal.

For any arbitrary transmit signal with FWESD $\mathcal{E}_s(F,K)$ it follows from (3) that the deflection coefficient is

$$d^2 = \sigma_A^2 \int_{-1/\lambda_0}^{1/\lambda_0} \int_{-W/2}^{W/2} \frac{\mathcal{E}_s(F,K)}{P_h(F,K)\mathcal{E}_s(F,K)+\sigma_n^2} dF dK \tag{16}$$

where $\int_{-1/\lambda_0}^{1/\lambda_0} \int_{-W/2}^{W/2} \mathcal{E}_s(F,K) dF dK = \mathcal{E}$ to satisfy the energy constraint. Both these deflection coefficients are shown in Figure 11 for $\sigma_A^2 = 1$ versus the energy-to-noise ratio $\mathcal{E}/\sigma_n^2$. It is seen that a substantial improvement in detectability is obtained by using careful signal design.

9 Discussion and Conclusions

We have derived an optimal detector and transmit signal for a radar that has to operate in a clutter environment. In contrast to a multitude of previous approaches the entire space-time problem is addressed. Consequently, the optimal signal is specified by its frequency-wavenumber energy spectral density. Subject to a certain type of clutter model the solution has been obtained analytically and an example given. The results indicate a substantial improvement in performance, as expected. However, the assumption that the optimal signal can be synthesized in practice is critical and will

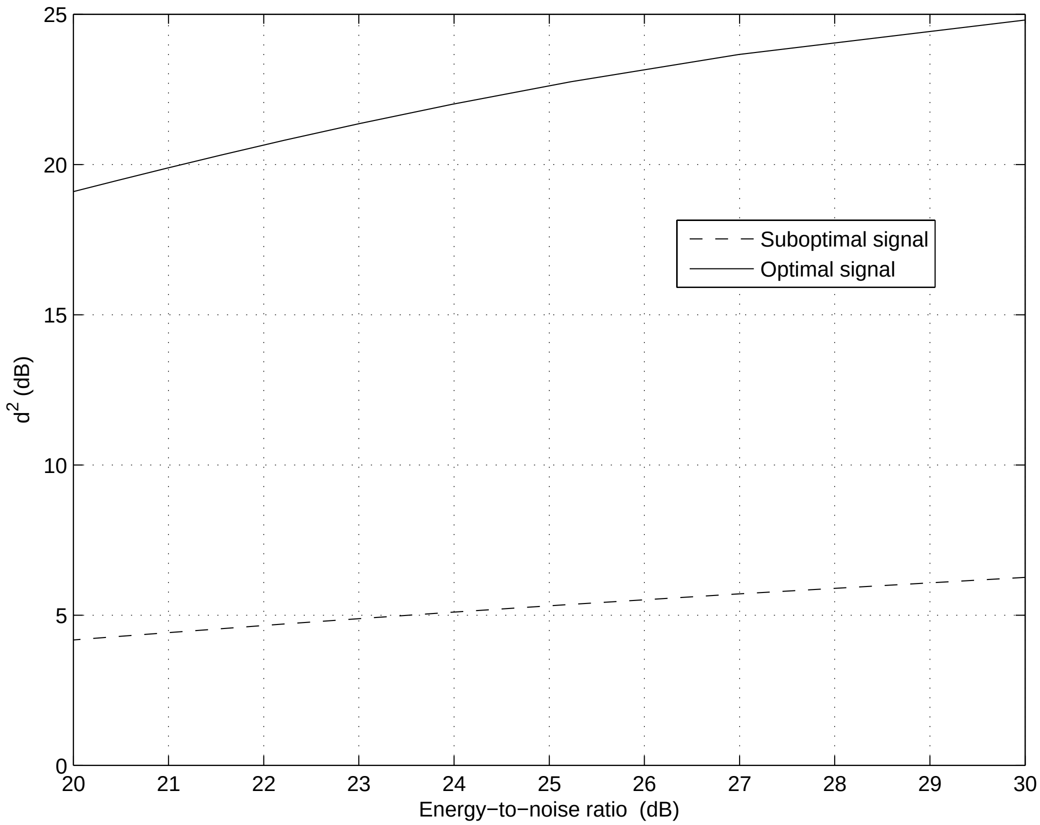

Figure 11: Deflection coefficients for the optimal signal and one that does not account for the clutter appropriately.

need further study to determine its feasibility. This is especially important in light of its space-time nature. Future studies will attempt to provide some real-world validation of the results contained herein.

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A Appendix A – Derivation of Neyman-Pearson Detector and its Performance

As per the assumptions described in Section 2 we consider the following detection problem.

$$\begin{aligned}\mathcal{H}_0 &: z(t,x) = c(t,x) + n(t,x)\\ \mathcal{H}_1 &: z(t,x) = r(t,x) + c(t,x) + n(t,x)\end{aligned}$$

for $-T/2 \leq t \leq T/2$ and $-L/2 \leq x \leq L/2$. The assumptions are that $r(t,x) = As_T(t,x) * *q(t,x)$, where $s_T(t,x) * *q(t,x)$ is a known complex signal, A is a complex random variable with $A \sim \mathcal{CN}(0, \sigma_A^2)$, $c(t,x)$ is a complex 2-D WSS Gaussian random process with zero mean and FWPSD $P_c(F,K) = P_h(F,K)LT|S_T(F,K)|^2 = P_h(F,K)\mathcal{E}_s(F,K)$, and $n(t,x)$ is a complex 2-D WSS Gaussian random process with zero mean and FWPSD $P_n(F,K)$. All signals and random processes are bandlimited to $W/2$ Hz and $1/\lambda_0$ cycles/m (see Figure 3). The random variable A, the random processes $c(t,x)$ and $n(t,x)$ are all independent of each other. We convert the received data into the frequency domain using a Fourier transform as defined by (9). For $WT > 16$ and $(2/\lambda_0)L > 16$ we can assert that the frequency samples $Z(F_m, K_n)$ for $F_m = m/T$, $m = -(M/2), \ldots, M/2$ and $K_n = n/L$, $n = -(N/2) \ldots, N/2$ are all independent [18]. They are also complex Gaussian random variables with zero mean and variance equal to the FWPSD value, which is $P_z(F_m, K_n)$. Hence, after Fourier transforming we obtain the $(M+1) \times (N+1)$ complex matrix

$$\mathbf{Z} = \begin{bmatrix} Z(F_{-M/2}, K_{-N/2}) & \ldots & Z(F_{-M/2}, K_{N/2}) \\ Z(F_{-M/2+1}, K_{-N/2}) & \ldots & Z(F_{-M/2+1}, K_{N/2}) \\ \vdots & \vdots & \vdots \\ Z(F_{M/2}, K_{-N/2}) & \ldots & Z(F_{M/2}, K_{N/2}) \end{bmatrix}$$

so that the equivalent detection problem is

$$\begin{aligned}\mathcal{H}_0 &: \mathbf{Z} = \mathbf{C} + \mathbf{N} = \mathbf{W}\\ \mathcal{H}_1 &: \mathbf{Z} = A\mathbf{S}_Q + \mathbf{C} + \mathbf{N} = A\mathbf{S}_Q + \mathbf{W}\end{aligned}$$

where all complex matrices are $(M+1) \times (N+1)$ matrices of Fourier transform samples. Also, we have let $S_Q(F,K) = \sqrt{LT} S_T(F,K) Q_t(F,K)$ and $W(F,K) = C(F,K) + N(F,K)$. Alternatively, we write the detection problem as

$$\begin{aligned}\mathcal{H}_0 &: Z[m,n] = W[m,n]\\ \mathcal{H}_1 &: Z[m,n] = AS_Q[m,n] + W[m,n]\end{aligned}$$

where $m = -(M/2), \ldots, M/2$, $n = -(N/2) \ldots, N/2$ and for example $Z[m,n] = Z(F_m, K_n)$ in order to simplify the notation and the following derivation. Note that all the $W[m,n]$ are independent complex Gaussian random variables with PDF $W[m,n] \sim \mathcal{CN}(0, P_w(F_m, K_n))$ and $A \sim \mathcal{CN}(0, \sigma_A^2)$ and is independent of the $W[m,n]$'s. Also, $S_Q[m,n]$ is known. To further simplify the notation let $\sigma_0^2[m,n] = P_w(F_m, K_n)$ so that $W[m,n] \sim \mathcal{CN}(0, \sigma_0^2[m,n])$ and $\sigma_0^2[m,n]$ indicates the variance under $\mathcal{H}_0$.

We next derive the Neyman-Pearson detector and its performance. First consider $\mathcal{H}_1$ and note that conditioned on knowing A the elements of $\mathbf{Z}$ are all independent. Thus, the complex conditional Gaussian PDF is

$$p(\mathbf{Z}|A;\mathcal{H}_1) = \prod_m \prod_n \frac{1}{\pi \sigma_0^2[m,n]} \exp\left[-\frac{|Z[m,n] - AS_Q[m,n]|^2}{\sigma_0^2[m,n]}\right].$$

The unconditional PDF is just the expected value of this or $p(\mathbf{Z};\mathcal{H}_1) = E_A[p(\mathbf{Z}|A;\mathcal{H}_1)]$ and the PDF under $\mathcal{H}_0$ is $p(\mathbf{Z}|0;\mathcal{H}_1)$. As a result the likelihood ratio is

$$L(\mathbf{Z}) = \frac{p(\mathbf{Z};\mathcal{H}_1)}{p(\mathbf{Z};\mathcal{H}_0)} = E_A\left[\underbrace{\frac{p(\mathbf{Z}|A;\mathcal{H}_1)}{p(\mathbf{Z};\mathcal{H}_0)}}_{L(\mathbf{Z}|A)}\right].$$

Now we have that

$$\begin{aligned}
L(\mathbf{Z}|A) &= \prod_m \prod_n \exp\left[-\frac{|Z[m,n] - AS_Q[m,n]|^2 - |Z[m,n]|^2}{\sigma_0^2[m,n]}\right] \\
&= \exp\left[-\sum_m \sum_n \left(\frac{|Z[m,n] - AS_Q[m,n]|^2 - |Z[m,n]|^2}{\sigma_0^2[m,n]}\right)\right]
\end{aligned}$$

Letting $Y[m,n] = Z[m,n]/\sigma_0[m,n]$ and $G[m,n] = S_Q[m,n]/\sigma_0[m,n]$ we have

$$L(\mathbf{Z}|A) = \exp\left[-\sum_m \sum_n |Y[m,n] - AG[m,n]|^2 - |Y[m,n]|^2\right].$$

Next using the identity

$$\mathrm{tr}(\mathbf{A}^H \mathbf{B}) = \sum_m \sum_n [\mathbf{A}]_{mn}^* [\mathbf{B}]_{mn}$$

we have

$$\begin{aligned}
L(\mathbf{Z}|A) &= \exp\left[-\mathrm{tr}[(\mathbf{Y} - A\mathbf{G})^H (\mathbf{Y} - A\mathbf{G})] + \mathrm{tr}[\mathbf{Y}^H \mathbf{Y})]\right] \\
&= \exp\left[-\mathrm{tr}(-\mathbf{Y}^H A\mathbf{G} - A^* \mathbf{G}^H \mathbf{Y} + |A|^2 \mathbf{G}^H \mathbf{G})\right] \\
&= \exp\left[-|A|^2 \mathrm{tr}(\mathbf{G}^H \mathbf{G}) + A\mathrm{tr}(\mathbf{Y}^H \mathbf{G}) + A^* \mathrm{tr}(\mathbf{G}^H \mathbf{Y})\right].
\end{aligned}$$

Next let $a = \mathrm{tr}(\mathbf{G}^H \mathbf{G})$, which is real, and $b = \mathrm{tr}(\mathbf{G}^H \mathbf{Y})$ so that

$$L(\mathbf{Z}|A) = \exp\left[-a|A|^2 + b^* A + bA^*\right]$$

and taking the expected value

$$L(\mathbf{Z}) = E_A[L(\mathbf{Z}|A)] = \int \exp\left[-(a|A|^2 - b^*A - bA^*)\right] \frac{1}{\pi\sigma_A^2} \exp\left[-\frac{|A|^2}{\sigma_A^2}\right] dA.$$

Completing the square we have

$$\begin{aligned} Q(A) &= |A|^2 \left(\underbrace{a + 1/\sigma_A^2}_{1/\sigma^2}\right) - b^*A - bA^* \\ &= \frac{1}{\sigma^2}\left(|A|^2 - b^*\sigma^2 A - b\sigma^2 A^*\right) \\ &= \frac{1}{\sigma^2}|A - b\sigma^2|^2 - |b|^2\sigma^2 \end{aligned}$$

and therefore

$$\begin{aligned} L(\mathbf{Z}) &= \frac{\pi\sigma^2}{\pi\sigma_A^2} \exp(|b|^2\sigma^2) \underbrace{\int \frac{1}{\pi\sigma^2} \exp\left[-(1/\sigma^2)|A - b\sigma^2|^2\right] dA}_{=1} \\ &= \frac{\sigma^2}{\sigma_A^2} \exp(|b|^2\sigma^2). \end{aligned}$$

Finally, we decide $\mathcal{H}_1$ if $\ln L(\mathbf{Z}) > \gamma'$ or if

$$\ln \frac{\sigma^2}{\sigma_A^2} + \sigma^2|b|^2 > \gamma'$$

or if

$$|b|^2 > \gamma.$$

But

$$\begin{aligned} |b|^2 &= |\mathrm{tr}(\mathbf{G}^H\mathbf{Y})|^2 = \left|\sum_m \sum_n \frac{Z[m,n]S_Q^*[m,n]}{\sigma_0^2[m,n]}\right|^2 \\ &= \left|\sum_m \sum_n \frac{Z(F_m,K_n)S_Q^*(F_m,K_n)}{P_w(F_m,K_n)}\right|^2 \\ &= \left|\sum_{m=-M/2}^{M/2} \sum_{n=-N/2}^{N/2} \frac{Z(F_m,K_n)S_T^*(F_m,K_n)\sqrt{LT}Q_t^*(F_m,K_n)}{P_h(F_m,K_n)LT|S_T(F_m,K_n)|^2 + P_n(F_m,K_n)}\right|^2 \end{aligned}$$

which is just (2).

To find the detection performance first note that we decide $\mathcal{H}_1$ if $|X|^2 > \gamma$, where $X = \mathrm{tr}(\mathbf{G}^H\mathbf{Y})$ and all the elements in the random matrix $\mathbf{Y}$ are complex Gaussian random variables. Hence, X is a linear combination of these elements and so is also a complex Gaussian random variable. As all means are zero we have that $X \sim \mathcal{CN}(0,\sigma_0^2)$ under $\mathcal{H}_0$ and $X \sim \mathcal{CN}(0,\sigma_1^2)$. This is a standard problem that we previously addressed in [1]. It was shown there that

$$P_D = P_{FA}^{\frac{1}{1+d^2}}$$

where P_D, P_{FA} denote the probability of detection and false alarm, respectively, and the deflection coefficient is

$$d^2 = \frac{\sigma_1^2 - \sigma_0^2}{\sigma_0^2}.$$

Hence, we need only find σ_0^2 and σ_1^2 or equvialently we require $E[|X|^2]$ under $\mathcal{H}_0$ and $\mathcal{H}_1$. Again reverting to the simpler notation we have

$$X = \sum_m \sum_n Y[m,n] G^*[m,n].$$

Under $\mathcal{H}_0$ $Y[m,n] = W[m,n]/\sigma_0[m,n]$, which has second moment equal to one and all samples are independent. Under $\mathcal{H}_1$ $Y[m,n] = AS_Q[m,n]/\sigma_0[m,n] + W[m,n]/\sigma_0[m,n]$, which has a second moment that we can readily calculate. Considering $\mathcal{H}_0$ first we have

$$\begin{aligned} E[|X|^2] &= E\left[\left|\sum_m \sum_n Y[m,n]G^*[m,n]\right|^2\right] \\ &= \sum_k \sum_l \sum_m \sum_n \underbrace{E[Y[k,l]Y^*[m,n]]}_{\delta_{k-m,n-l}} G^*[k,l]G[m,n] \\ &= \sum_m \sum_n |G[m,n]|^2 \end{aligned}$$

so that

$$\sigma_0^2 = \sum_m \sum_n \frac{|S_Q[m,n]|^2}{\sigma_0^2[m,n]}.$$

Next under $\mathcal{H}_1$

$$\begin{aligned} X &= \sum_m \sum_n \frac{Z[m,n]S_Q^*[m,n]}{\sigma_0^2[m,n]} \\ &= \sum_m \sum_n \frac{(AS_Q[m,n] + W[m,n])S_Q^*[m,n]}{\sigma_0^2[m,n]} = \sum_m \sum_n \left(Y[m,n] + A\frac{S_Q[m,n]}{\sigma_0[m,n]}\right) G^*[m,n]. \end{aligned}$$

Therefore,

$$\begin{aligned} E[|X|^2] &= E\left[\left|\sum_m \sum_n \left(Y[m,n] + A\frac{S_Q[m,n]}{\sigma_0[m,n]}\right) G^*[m,n]\right|^2\right] \\ &= \sum_k \sum_l \sum_m \sum_n E\left[\left(Y[k,l] + A\frac{S_Q[k,l]}{\sigma_0[k,l]}\right)\left(Y[m,n] + A\frac{S_Q[m,n]}{\sigma_0[m,n]}\right)^*\right] G^*[k,l]G[m,n] \\ &= \sum_k \sum_l \sum_m \sum_n \left(\delta_{k-m,n-l} + \sigma_A^2 \frac{S_Q[k,l]S_Q^*[m,n]}{\sigma_0[k,l]\sigma_0[m,n]}\right) G^*[k,l]G[m,n] \\ &= \sigma_0 + \sigma_A^2 \left|\sum_m \sum_n \frac{S_Q^*[m,n]G[m,n]}{\sigma_0[m,n]}\right|^2 \\ &= \sigma_0^2 + \sigma_A^2 \left(\sum_m \sum_n \frac{|S_Q[m,n]|^2}{\sigma_0^2[m,n]}\right)^2. \end{aligned}$$

As a result we have that

$$
\begin{aligned}
d^2 &= \sigma_A^2 \sum_m \sum_n \frac{|S_Q[m,n]|^2}{\sigma_0^2[m,n]} \\
&= \sigma_A^2 \sum_m \sum_n \frac{|S_Q(F_m,K_n)|^2}{P_h(F_m,K_n)LT|S_T(F_m,K_n)|^2 + P_n(F_m,K_n)}
\end{aligned}
$$

and as an approximation since $F_m = m/T$ and $K_n = n/L$ we have that

$$
\begin{aligned}
d^2 &= \sigma_A^2 LT \sum_{m=-M/2}^{M/2} \sum_{n=-N/2}^{N/2} \frac{|S_Q(F_m,K_n)|^2}{P_h(F_m,K_n)LT|S_T(F_m,K_n)|^2 + P_n(F_m,K_n)} \frac{1}{T}\frac{1}{L} \\
&\approx \sigma_A^2 \int_{-1/\lambda_0}^{1/\lambda_0} \int_{-W/2}^{W/2} \frac{LT|S_T(F,K)|^2 LT|Q_t(F,K)|^2}{P_h(F,K)LT|S_T(F,K)|^2 + P_n(F,K)} dF dK
\end{aligned}
$$

which is (10).