A Poisson Spectral Representation for Random Process Modeling

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Abstract

In a previous paper it was shown that a wide sense stationary random process could be represented as a sum of sinusoids with random frequencies. We provide an extension of that theory by representing the process as a marked nonhomogeneous Poisson process in frequency. This leads to a new spectral representation with some interesting properties. A realization of the random process can be synthesized very simply by generating a realization of the Poisson process. The first-order probability density function is that of a compound Poisson random variable, a commonly used model for clutter, as for example in the Middleton Class A probability density function. Because of its generality the extension to a multidimensional power spectral density is immediate and an example is given of that extension. Applications to spectral hypothesis testing are also described.

1 Introduction

The spectral representation for a wide sense stationary (WSS) random process is of utmost importance in modern signal processing theory and practice [2]. It forms the basis for the the science (and sometimes "art") of spectral estimation [7]. This representation relies on summing together a set of sinusoids of fixed frequencies but random amplitudes and random phases. Alternatively, one

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can view the complex amplitude in the context of a random spectral measure, which is uncorrelated over disjoint frequency sets. Previously, it was shown that by also assuming the frequencies to be random, one could obtain a representation that allowed the independent specification of the power spectral density (PSD) and the first-order probability density function (PDF) [11]. The only proviso was that the PDF had to be in the class of infinitely divisible densities [4], which includes many well known PDFs such as the Gaussian, Cauchy, K-PDF, etc. This representation had the advantage of lending itself nicely to generation of realizations of the random process without the difficulties usually caused by coupling of the PSD to the PDF.

In this paper we show that an extension of this previously reported representation is obtained by allowing the random frequencies, which previously were assumed to be independent and identically distributed (IID) random variables, to instead be given by a realization of a nonhomogenous Poisson process in *frequency*. The previously reported results then are obtained if we condition on the number of frequency events in the given frequency band. The PSD, as before, is easily specified by the intensity of the Poisson process and the first-order PDF is that of a compound Poisson random variable. The latter comprises a broad class of PDFs with the Gaussian being a limiting case.

Besides being of theoretical interest the proposed representation has many practical applications. Modeling of clutter spectra and nonGaussian PDFs is one area of interest [6, 1]. Another is the use of the nonhomogeneous Poisson process as a model for neural responses, with an example being the output of the auditory nerve [18]. Synthesizing realizations of a random process to approximate a desired PSD is greatly simplified since there is no need to design coloring filters. For example, the Gaussian PSD is frequently used as a model but is difficult to synthesize using a rational transfer function model [7]. No such limitation is present if the frequencies are assumed to be random. Finally, the representation is very general due to the generality of the Poisson process [13]. In particular, the extensions to the multidimensional case, such as required for two- and three-dimensional PSDs, is straightforward. It should also be mentioned that for deterministic signals, a similar nonuniformly spaced frequency representation is available [12].

In summary, the utility of the new representation is the following:

1. A realization of a WSS random process with any PSD (either one-dimensional or multidimensional) can be easily generated. No spectral factorization or covariance matrix square rooting is necessary.

2. Correlated random processes with a nonGaussian PDF, either a compound Poisson or more generally an infinitely divisible PDF, can be represented.

3. The model more closely represents typical scattering phenomena such as reverberation, clutter, and target returns, when the number of scatterers is small.

4. The model may be useful in explaining neural responses, which are two-dimensional such as in the auditory nerve or optic nerve.

5. The ensemble autocorrelation sequence and PSD are easily specified. Therefore, the representation may be useful in theoretical work.

6. Extensions to the multichannel random process and evolutionary random processes is possible.

The content of the paper is as follows. In Section 2 we describe the spectral representation while Section 3 summarizes its properties. In Section 4 an explicit example is given. Section 5 includes some typical applications. A summary and discussion is included in Section 6. All derivations are relegated to several appendices.

2 The Poisson Spectral Representation

The background for this section can be found in [13]. For a discrete-time wide sense stationary random process it is well known that the process has the *spectral representation* [2]

$$X[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} \exp(j2\pi f n) X(df) \tag{1}$$

where $X(df)$ is the complex *spectral measure* and can be viewed as a complex random variable that gives the amplitude and phase of the sinusoid whose frequency is at f. The random variable has a zero mean and is uncorrelated for different frequency sinusoids. In our Poisson representation it will be necessary to assume that not only are the sinusoidal complex amplitudes uncorrelated but they are also independent. Of course if the random process were Gaussian, the representation of (1) would also imply independence. The power at frequency f is given by $E[|X(df)|^2]$ and hence the PSD is just $P_X(f) = E[|X(df)|^2]/df$. In the proposed Poisson Spectral Representation (PSR) we simplify the discussion by considering only a *real* random process and thus, use the real representation

$$X[n] = \frac{1}{\sqrt{\lambda_0/2}} \sum_{k=1}^{N} A_k \cos(2\pi F_k n + \Phi_k) \qquad -\infty < n < \infty \tag{2}$$

where $\{A_1, A_2, \ldots, A_N\}$ are IID positive amplitude random variables, $\{\Phi_1, \Phi_2, \ldots, \Phi_N\}$ are IID phase random variables uniformly distributed on $[0, 2\pi)$, and with the amplitudes independent of the phases. The number of sinusoids N is a Poisson random variable with mean λ_0 and the frequencies $\{F_1, F_2, \ldots, F_N\}$ are the *point events in frequency* of a *nonhomogeneous Poisson random process* on the interval $0 \leq f \leq 1/2$. The Poisson random process N is independent of the amplitudes and phases. In contrast to the usual spectral representation of (1) where the frequencies are fixed, and usually chosen in the limit as *uniformly spaced* in frequency, in the PSR the frequencies are randomly distributed throughout the frequency interval as a nonhomogeneous Poisson process. This representation then gives rise to a given PSD via a nonuniform distribution of sinusoidal frequency components.

The representation of (2) in which we sum a function evaluated at the points of a Poisson process and for which the function also depends upon the outcomes of other random variables is called a *marked Poisson process*. It can alternatively be written as

$$X[n] = \sum_{k=1}^{N} g_n(F_k, (A_k, \Phi_k)) \tag{3}$$

where

$$g_n(F, (A, \Phi)) = \frac{1}{\sqrt{\lambda_0/2}} A \cos(2\pi F n + \Phi).$$

In this case, the random variable associated with the kth frequency event of the Poisson process is (A_k, Φ_k) and is called the "mark". It is independent of the marks at the other frequencies and also of the other frequency events. As such, it produces a multidimensional Poisson process denoted by N with events as depicted in Figure 1.

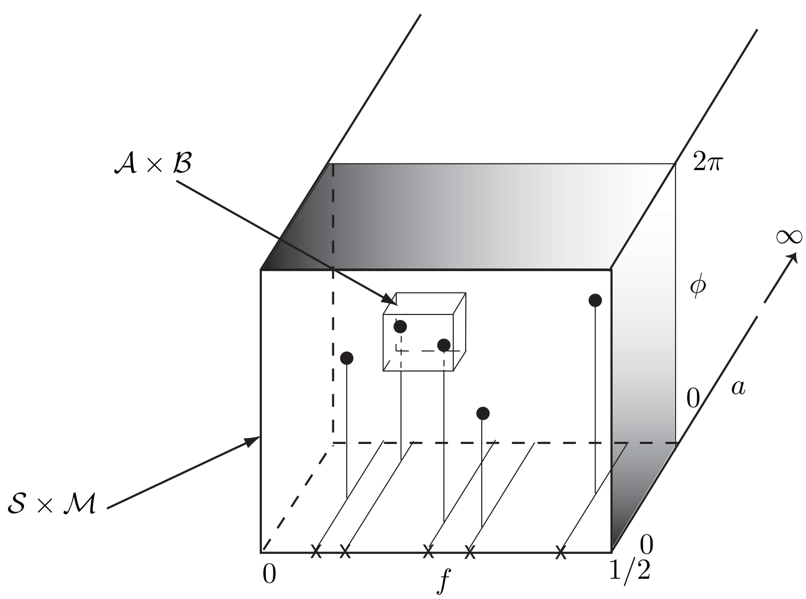

Figure 1: Illustration of an outcome of a marked Poisson process N (shown as solid dots) with event F (shown as x's) and mark (A, Φ). Note that the number of events in $\mathcal{A} \times \mathcal{B}$ is two for the pictured realization and the average number of events is the mean measure $\mu(\mathcal{A} \times \mathcal{B})$.

The space in which the marked events occur will be denoted by $\mathcal{S} \times \mathcal{M}$. Here $\mathcal{S}$ denotes the frequency interval $0 \le f \le 1/2$, where the frequency resides and $\mathcal{M}$ denotes the marked space $[0, \infty) \times [0, 2\pi)$, where the amplitude and phase reside. The number of events in a subset V of this space is given by $N(V)$, which is a Poisson random variable with mean *measure* $E[N(V)] = \mu(V)$. Also, the number of events in disjoint subsets are independent of each order, according to the

Poisson assumption. Since a projection of a marked Poisson process is also a Poisson process, the frequencies form a nonhomogenous Poisson process with mean measure denoted by

$$\mu([f_1, f_2] \times \mathcal{M}) = E[N([f_1, f_2] \times \mathcal{M})] = \int_{f_1}^{f_2} \lambda(f) df$$

where we have assumed that all the measures are absolutely continuous. We interpret this assumption as saying that the Poisson process in frequency is nonhomogeneous and has an *intensity* of $\lambda(f)$. Thus the frequency realization is such that there are more events in frequency intervals where $\lambda(f)$ is large, i.e, where the "arrival rate" or intensity is large. Overall, it can be shown that the mean measure for the Poisson process N for an arbitrary volume $\mathcal{A} \times \mathcal{B}$, where $\mathcal{A} \in \mathcal{S}$ and $\mathcal{B} \in \mathcal{M}$ is

$$\mu(\mathcal{A} \times \mathcal{B}) = \int_{\mathcal{A} \times B} \lambda(f) p_A(a) p_\Phi(\phi) d\phi \, da \, df \tag{4}$$

where $p_A(a)$ and $p_\Phi(\phi)$ are the PDFs of the amplitude and phase, respectively. We normalize the intensity by letting

$$\lambda(f) = \lambda_0 p(f)$$

where $\int_0^{1/2} p(f) df = 1$ so that $p(f)$ can be interpreted as a PDF in frequency. With this normalization the total number of expected events in $\mathcal{S} \times \mathcal{M}$ is

$$\mu(\mathcal{S} \times \mathcal{M}) = \int_0^{1/2} \int_0^\infty \int_0^{2\pi} p_\Phi(\phi) p_A(a) \lambda_0 p(f) d\phi \, da \, df = \lambda_0.$$

With these assumptions the PSR can be written as

$$X[n] = \int_{\mathcal{S} \times \mathcal{M}} \frac{a}{\sqrt{\lambda_0/2}} \cos(2\pi f n + \phi) N(df \times (da, d\phi)) \tag{5}$$

where $N(\mathcal{A} \times \mathcal{B})$ is the number of events occuring in the "rectangle" $\mathcal{A} \times \mathcal{B}$. This is similar to (1) except for the mark (A, Φ) and the property that the random counting measure, i.e., random variable N, is Poisson and is independent, and not just uncorrelated, in nonoverlapping sets in $\mathcal{S} \times \mathcal{M}$. We can alternatively think of the random counting measure as

$$N(df \times (da, d\phi)) = \sum_{k=1}^{N(\mathcal{S} \times \mathcal{M})} \delta(f - f_k, a - a_k, \phi - \phi_k) df \, da \, d\phi$$

where δ is a three-dimensional Dirac delta function. Inserting this into (5) will produce (2). Note that if $N(\mathcal{S} \times \mathcal{M}) = 0$ then we define $X[n]$ as zero, although this will be a low probability occurrence, especially for large λ_0. The assumption of large λ_0 is desirable in that the $X[n]$ process will be shown to be ergodic in the autocorrelation sequence only as $\lambda_0 \to \infty$, and is necessary for a practical representation. We next state the properties of (2) or equivalently (5) with the derivations given in the Appendices.

3 Properties of the Representation

The $X[n]$ random process as defined by (2) or equivalently by (5) with its accompanying assumptions can be shown to possess the following properties (see the Appendices for the derivation):

1. The process is zero mean, i.e., $E[X[n]] = 0$ for $-\infty < n < \infty$. This is due to the assumption that the phase is uniformly distributed.

2. The process possesses an autocorrelation sequence, which together with the first property, shows that it is WSS.

3. The PSD is given by

$$P_X(f) = \frac{E[A^2]}{2} p(|f|) \qquad -1/2 \leq f \leq 1/2. \tag{6}$$

Thus, the PSD is specified by choosing the intensity of the nonhomogeneous Poisson process in frequency since $\lambda(f) = \lambda_0 p(f)$ on the interval $0 \leq f \leq 1/2$. The total power is seen to be $E[X^2[n]] = E[A^2]$, and is independent of λ_0 (the reason for the chosen normalization in (2) of $\sqrt{\lambda_0/2}$).

4. The process is ergodic in the mean. This is due to the assumption of an absolutely continuous spectral measure since this implies an absence of a delta function in the PSD at $f = 0$ [2].

5. The process is ergodic in the autocorrelation sequence as $\lambda_0 \to \infty$. This is reasonable in that as the intensity of the Poisson process in frequency increases, more frequency events occur and hence a more complete temporal description of the process is obtained.

6. The first-order PDF of the process does not depend on n so that the process is stationary to first-order, and is given by the PDF of a compound Poisson random variable as

$$X[n] = \sum_{k=1}^{N} U_k \tag{7}$$

where N is a Poisson random variable with mean λ_0 and the U_k's are IID random variables and specifically $U_k = A_k \cos(\Phi_k)$. The characteristic function of $X[n]$ is

$$\psi_{X[n]}(\omega) = E[\exp(j\omega X[n])] = \exp\left[\lambda_0 \left(\int_0^\infty J_0\left(\frac{\omega a}{\sqrt{\lambda_0/2}}\right) p_A(a)da - 1\right)\right] \tag{8}$$

where J_0 is the Bessel function of order zero. It can be shown that as $\lambda_0 \to \infty$, $X[n]$ becomes Gaussian with zero mean and variance $E[A^2]$, and this holds independently of the PDF of A. Other first-order PDFs can be constructed by appealing to the properties of the *conditional* Poisson process as explained in Section 5. Furthermore, as $\lambda_0 \to \infty$, the process becomes a Gaussian random process in a similar fashion to the well known convergence property of a filtered Poisson process as the intensity becomes large [16].

4 An Example

Assume that we wish to represent a random process whose first-order PDF is that of a compound Poisson-Gaussian random variable. By the latter we mean that the random variable can be represented as

$$X = \sum_{k=1}^{N} U_k \tag{9}$$

where the U_k's are IID Gaussian random variables with mean zero and variance σ^2/λ_0 and N is a Poisson random variable with mean λ_0 and independent of the U_k's. The PDF can be written as

$$p_X(x) = \sum_{k=0}^{\infty} \exp(-\lambda_0)\frac{\lambda_0^k}{k!}\mathcal{N}\left(0, k\frac{\sigma^2}{\lambda_0}\right)$$

where

$$\mathcal{N}(0,\sigma_1^2) = p_U(u) = \frac{1}{\sqrt{2\pi\sigma_1^2}}\exp[-(1/(2\sigma_1^2))u^2] \tag{10}$$

and $\mathcal{N}(0,0)$ is defined to be zero. This is easily verified from (9) by finding the PDF of the sum by first conditioning on N and then unconditioning by summing with respect to the probability mass function of the Poisson random variable. To determine the PDF necessary for A (recall that $U_k = A_k\cos(\Phi_k)$ with $\Phi_k \sim \mathcal{U}([0,2\pi))$ we note that for a compound Poisson random variable the characteristic function is [4]

$$\psi_X(\omega) = \exp\left[\lambda_0(\psi_U(\omega) - 1)\right]$$

so that from (8)

$$\psi_U(\omega) = \int_0^{\infty} J_0\left(\frac{\omega a}{\sqrt{\lambda_0/2}}\right) p_A(a)da.$$

Since $U \sim \mathcal{N}(0, \sigma^2/\lambda_0)$

$$\psi_U(\omega) = \exp\left[-\frac{1}{2}\frac{\sigma^2}{\lambda_0}\omega^2\right]$$

and therefore we require the solution of

$$\exp\left[-\frac{1}{2}\frac{\sigma^2}{\lambda_0}\omega^2\right] = \int_0^{\infty} J_0\left(\frac{\omega a}{\sqrt{\lambda_0/2}}\right) p_A(a)da$$

for $p_A(a)$. But for $\alpha > 0$ it can be shown that [5]

$$\alpha\int_0^{\infty} J_0(\beta x)x\exp\left(-\frac{1}{2}\alpha x^2\right)dx = \exp\left[-\frac{1}{2}\beta^2/(2\alpha)\right]$$

and letting $x = a$, $\beta = \omega/\sqrt{\lambda_0/2}$ and $\alpha = 1/\sigma^2$, we have

$$\frac{1}{\sigma^2}\int_0^{\infty} J_0\left(\frac{\omega}{\sqrt{\lambda_0/2}}a\right) a\exp\left(-\frac{1}{2}a^2/\sigma^2\right)da = \exp\left[-\frac{1}{2}\frac{\omega^2\sigma^2}{\lambda_0}\right]$$

so that

$$p_A(a) = \frac{a}{\sigma^2} \exp\left(-\frac{1}{2}\frac{a^2}{\sigma^2}\right)$$

which is recognized as a Rayleigh PDF. Thus, the amplitude and phase random variables may be generated by generating two independent Gaussian random variables (x, y) each with PDF $\mathcal{N}(0, \sigma^2)$ after conversion to a magnitude and phase, thus forming $a = \sqrt{x^2 + y^2}$ and $\phi = \arctan(y/x)$. This result is noted from (2) since for the amplitude and phase to be independent with Rayleigh and uniform PDFs, respectively, $X = A\cos(\Phi)$ and $Y = -A\sin(\Phi)$ must be independent and both Gaussian [10].

The PSD is easily specified from (6) so that the intensity of the Poisson process in frequency should be chosen as

$$\lambda(f) = \lambda_0 p(f) = \frac{2\lambda_0 P_X(f)}{E[A^2]} = \lambda_0 \frac{P_X(f)}{\sigma^2} \quad 0 \leq f \leq 1/2.$$

To actually generate a nonhomogeneous Poisson process with this intensity one can use the methods described in [18]. For example, a *homogeneous* Poisson process is easily generated by generating the interevent times as independent exponential random variables. Then, the realization is transformed by a nonlinear transformation obtained from $\lambda(f)$ to obtain the desired *nonhomogeneous* Poisson process realization.

5 Rapprochement with Previous Results

The previous representation used in [11] was

$$X[n] = \frac{1}{\sqrt{M/2}} \sum_{i=1}^{M} A_i \cos(2\pi F_i n + \Phi_i)$$

where the random variables (A_i, Φ_i) were IID with the amplitudes and phases independent of each other, and the phases were uniformly distributed. It was assumed that M is a given *constant*. These are nearly the same assumptions as for (1). The difference lies with the assumption on the frequency random variables. Previously, these were IID with PDF $p_F(f)$ on the interval $[0, 1/2]$ and independent of the amplitude and phase random variables. In the PSR we consider the frequencies as random events distributed according to a nonhomogeneous Poisson random process with intensity $\lambda(f) = \lambda_0 p(f)$. However, if in the PSR model we *fix* the number of events N as the constant M, or equivalently condition on the number of frequency events in $[0, 1/2]$, then the PSR reduces to our previous model. This is a well known result that a nonhomogeneous Poisson process with intensity $\lambda(f)$, conditioned on the number of events, has the same distribution as the *order statistics* of M IID random variables with the PDF [18]

$$p_F(f) = \frac{\lambda(f)}{\int_0^{1/2} \lambda(f) df} \qquad 0 \leq f \leq 1/2.$$

But for the PSR $\lambda(f) = \lambda_0 p(f)$ so that

$$p_F(f) = \frac{\lambda_0 p(f)}{\int_0^{1/2} \lambda_0 p(f) df} = p(f)$$

since $\int_0^{1/2} p(f)df = 1$. Hence, if we condition on the number of frequency events in $[0, 1/2]$, then $X[n]$ has the properties listed in [11]. The only difference is that by conditioning the characteristic function of $X[n]$ becomes

$$\left(\psi_{x[n]}(\omega|N = M)\right)^M = \left(\int_0^\infty J_0\left(\frac{\omega a}{\sqrt{M/2}}\right) p_A(a)da\right)^M.$$

If we were now to assume that $N \sim \text{Poiss}(\lambda_0)$, then by taking the expected value of $\left(\psi_{x[n]}(\omega|N = M)\right)^M$ with respect to a Poisson random variable with mean λ_0, we would recover the characteristic function of (8).

6 Some Applications

6.1 Class A Noise with Arbitrary PSD

Middleton's class A noise model has been found to accurately predict scattering phenomena in radar, sonar, and communications [15]. The first-order PDF is defined as

$$p_{X[n]}(x; \lambda_0, \Omega, \sigma_G^2) = \exp(-\lambda_0) \sum_{k=0}^{\infty} \frac{\lambda_0^k}{k!} \mathcal{N}(0, \sigma_k^2) \tag{11}$$

where $\mathcal{N}(0, \sigma^2)$ denotes a Gaussian PDF with mean zero and variance σ^2 and $\sigma_k^2 = k(\Omega/\lambda_0) + \sigma_G^2$. The PDF corresponds to the sum of a compound Poisson random variable, which represents scattering such as is encountered in reverberation or clutter, and a Gaussian random variable, which represents ambient noise. As such, the sum random variable can be expressed as

$$X[n] = \sum_{k=1}^{N} U_k + W$$

where $N \sim \text{Poiss}(\lambda_0)$, $U_k \sim \mathcal{N}(0, \Omega/\lambda_0)$ and are IID, and $W \sim \mathcal{N}(0, \sigma_G^2)$ and all random variables are independent of each other. The total process power is $E[X^2[n]] = \Omega + \sigma_G^2$. Referring to (11) we see that the nonGaussian part can alternatively be represented by a PSR with $E[A^2] = \Omega$. To this we can add white Gaussian noise with variance σ_G^2 to represent the ambient noise. This follows because the PDF of the nonGaussian (NG) part is

$$p_{NG}(x; \lambda_0, \Omega) = \exp(-\lambda_0) \sum_{k=0}^{\infty} \frac{\lambda_0^k}{k!} \mathcal{N}\left(0, k\frac{\Omega}{\lambda_0}\right)$$

and is convolved with the PDF of the Gaussian part

$$p_G(x;\sigma_G^2) = \frac{1}{\sqrt{2\pi\sigma_G^2}} \exp\left(-\frac{1}{2\sigma_G^2}x^2\right)$$

which by linearity of convolution produces (11).

The PSD of $X[n]$ becomes upon noting the independence and hence uncorrelateness of the nonGaussian and Gaussian parts

$$P_X(f) = P_{NG}(f) + \sigma_G^2.$$

Here $P_{NG}(f)$ is the PSD of the nonGaussian part and is realized using (6). The final representation is

$$X[n] = \frac{1}{\sqrt{\lambda_0/2}} \sum_{k=1}^{N} A_k \cos(2\pi F_k n + \Phi_k) + W[n] \tag{12}$$

where N is Poisson with intensity $\lambda(f) = \lambda_0 P_{NG}(f)/\Omega$, A_k's are IID Rayleigh random variables with parameter Ω, Φ's are IID uniform random variables on $[0, 2\pi)$, and $W[n]$ is white Gaussian noise with variance σ_G^2.

6.2 Generation of Multidimensional WSS Random Process Realizations

It is frequently desired to generate a realization of an m-dimensional Gaussian random process with a given PSD. A common procedure is to filter an m-dimensional white Gaussian noise process using a "coloration" filter. As an alternative, the use of the PSR provides a more direct method. We illustrate for a two-dimensional WSS process with the extension to any number of dimensions being obvious.

The representation becomes

$$X[n_1, n_2] = \frac{1}{\sqrt{\lambda_0/2}} \sum_{k=1}^{N} A_k \cos[2\pi(F_{1_k} n_1 + F_{2_k} n_2) + \Phi_k] \tag{13}$$

where the A_k's are IID Rayleigh random variables, the Φ_k's are uniform random variables with the amplitudes and phases independent of each other, as for the one-dimensional case. The only difference is in the marked Poisson process now being defined over the space $\mathcal{S} \times \mathcal{M}$, where $\mathcal{S} = [0, 1/2] \times [0, 1]$. A realization of the two-dimensional random process $X[n_1, n_2]$ now requires one to implement a two-dimensional nonhomogeneous Poisson process with intensity

$$\lambda(f_1, f_2) = \lambda_0 \frac{2P_X(f_1, f_2)}{E[A^2]} \qquad 0 \le f_1 \le 1/2;\ -1/2 \le f_2 \le 1.$$

If we condition on the number of events, then the generation process is simplified since the frequency events (F_{1_k}, F_{2_k}) becomes IID random vectors with joint PDF

$$p_{F_1,F_2}(f_1, f_2) = \frac{2P_X(f_1, f_2)}{E[A^2]} \qquad 0 \le f_1 \le 1/2;\ -1/2 \le f_2 \le 1.$$

Hence, the generation reduces to that of generating a two-dimensional random vector outcome with a given PDF. Note that any PDF may be approximated by a sum of two-dimensional Gaussian PDFs for which generation of Gaussian random vectors becomes a simple task. The same approach can also be used for m-dimensional PSDs that can be approximated by a sum of multivariate Gaussians.

6.3 Spectral Hypothesis Testing

We now describe an approach to discrimination of PSDs basd on the PSR. In the process of doing so, we point out some interesting correspondences between PDF metrics and PSD metrics. Assume that we wish to discriminate between two PSDs, with the extension to any number of PSDs being obvious. Since the PSD is related to the intensity in the PSR as

$$\lambda(f) = \frac{2\lambda_0 P_X(f)}{E[A^2]} \tag{14}$$

we can use the intensity for discrimination. To simplify the discussion we let the power in each process be the same, which for convenience is taken to be $r_x[0] = E[A^2] = 2$. Then from (14) we have that $\lambda(f) = \lambda_0 P_X(f)$. Since the marks of the Poisson process are independent of the frequency events and since $P_X(f)$ only depends on the intensity, we can base any decision on just the intensity realization. It can be shown that the part of the log-PDF that depends on the intensity is given by [18]

$$l = -\int_0^{1/2} \lambda(f)df + \int_0^{1/2} \ln \lambda(f) N(df) \tag{15}$$

and this becomes

$$\begin{aligned} l &= -\int_0^{1/2} \lambda_0 P_X(f)df + \int_0^{1/2} \ln\left(\lambda_0 P_X(f)\right) N(df) \\ &= -\lambda_0 + \frac{1}{2}\ln\lambda_0 + \int_0^{1/2} \ln P_X(f) N(df). \end{aligned} \tag{16}$$

Finally, assuming the possible PSDs are equally likely to occur we minimize the probability of decision error by choosing the PSD for which

$$l' = \int_0^{1/2} \ln P_X(f) N(df) = \sum_{k=1}^{N} \ln P_X(f_k)$$

is maximum. Usually, the frequency events are not observable but only $x[n]$ is observed. Some applications for which the frequency events can be observed are in neural auditory coding [17]. For the former case we proceed by noting that

$$E[N(df)] = \lambda(f)df = \lambda_0 P_X(f)df \approx \lambda_0 I(f)df$$

where $I(f)$ is the periodogram, which is given by

$$I(f) = \frac{1}{M} \left| \sum_{m=0}^{M-1} x[m] \exp(-j2\pi f m) \right|^2 .$$

The data set $x[m]$ for $m = 0, 1, \ldots, M-1$ is assumed to have been observed. Thus, we can use

$$l' = \int_0^{1/2} (\ln P_X(f)) \, \lambda_0 I(f) df.$$

Finally if we have two possible PSDs $P_{X_1}(f)$ and $P_{X_2}(f)$, we choose the one that maximizes over $i = 1, 2$

$$\xi_i = \int_0^{1/2} I(f) \ln P_{X_i}(f) df. \tag{17}$$

A computer simulation example is now presented.

We assume that we wish to discriminate between two Gaussian autoregressive (AR) random processes of order four with the PSDs shown in Figure 1. They ae assumed to be equally likely to occur. To do so we consider two approaches. The first is the one given above as (17). The second,

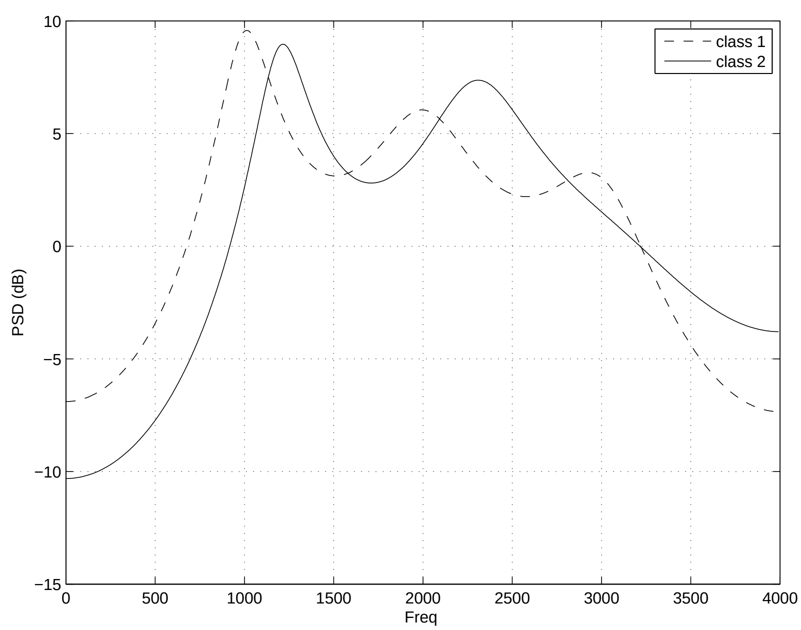

Figure 2: Two AR(4) random process PSDs to be classified.

which is used as a baseline, is the asymptotic log-PDF of a Gaussian random process [8]. It can be shown to be (apart from a constant not depending on the PSD)

$$\ln p = -\int_{-1/2}^{1/2} \left(\ln P_X(f) + \frac{I(f)}{P_X(f)} \right) df.$$

We decide upon the PSD that maximizes $\ln p$, i.e., we use the maximum likelihood (ML) decision rule [9]. The measure of performance is the probability of correct classification P_c. To demonstrate an apparent advantage of the Poisson statistic we corrupt the Gaussian process by adding IID Laplacian noise of varying powers. The results will then indicated a measure of robustness to modeling assumptions. In Figure 3 we plot the probability of correct classification versus signal-to-noise ratio (SNR), where the latter is defined as the Gaussian random process power divided by the Laplacian noise power. It is seen that when the modeling assumptions are satisfied, i.e.,

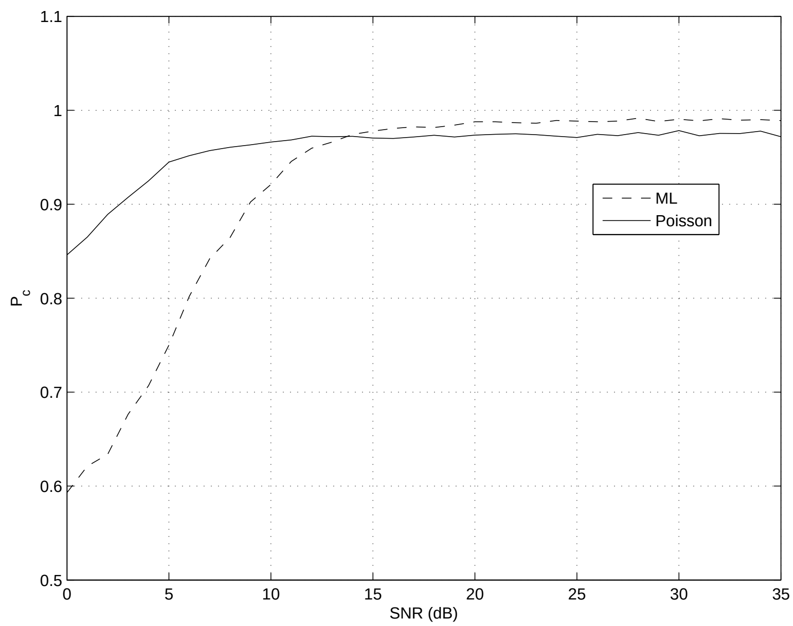

Figure 3: Probability of correct classification versur SNR.

no corrupting noise, that the Gaussian ML statistic produces slightly better results. This is as expected since the ML rule is the optimal approach, assuming the true PDF is used. However, when this is not the case, indicated by a worsening SNR, the Poisson test statistic significantly outperforms the Gaussian ML decision rule. It is not clear at this point why there is such an improvement in robustness so that further study is warranted. It should be mentioned that similar results for discrimation of deterministic energy spectral densities were reported upon in [12].

An interesting interpretation of the test statistic of (17) is next described. Recall that we have normalized the PSD so that

$$\int_0^{1/2} P_X(f)df = 1.$$

With this normalization the PSD integrates to one over the frequency interval $[0, 1/2]$ and so has the same mathematical properties as a PDF. Carrying the analogy further we also note that $E[I(f)] = P_X(f)$ [7]. If $i = 1$ is the correct PSD, then the difference of the expected values of the statistic of (17) is given by

$$\begin{aligned}
E_1[\xi_1] - E_2[\xi_2] &= \int_0^{1/2} E_1[I(f)] \ln P_{X_1}(f) df - \int_0^{1/2} E_1[I(f)] \ln P_{X_2}(f) df \\
&= \int_0^{1/2} P_{X_1}(f) \ln P_{X_1}(f) df - \int_0^{1/2} P_{X_1}(f) \ln P_{X_2}(f) df \\
&= \int_0^{1/2} P_{X_1}(f) \ln \frac{P_{X_1}(f)}{P_{X_2}(f)} df.
\end{aligned}$$

This is recognized as the Kullback-Liebler distance measure [14]. This measure had previously been used to distinguish speech-like signals based on their periodograms [12].

7 Conclusions

We have presented an alternative spectral representation of a wide sense stationary random process. It is based on the assumption that the frequencies are random variables, and in particular, they are the events of a nonhomogeneous Poisson process in frequency. The theoretical properties were derived and some applications given to illustrate its usefulness in practice. The results presented extend previous ones and so forms a more complete theory. Future work will undoubtedly add to the aforementioned applications.

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A Derivation of Mean and Covariance

Using the standard notation of Kingman [13], we let the Poisson process be denoted by Π and the marked Poisson process by Π^*. The event is denoted by the vector in R^3 as $\mathbf{x}$. Then we wish to determine the mean and covariance of

$$Z_m = \sum_{\mathbf{X}\in\Pi^*} g_m(\mathbf{X}) \tag{18}$$

It is shown in Appendix C that the first four moments are given as

$$E[Z_m] = \int_{\mathcal{S}\times\mathcal{M}} g_m(\mathbf{x})\mu(d\mathbf{x}) \tag{19}$$

$$E[Z_m Z_n] = \int_{\mathcal{S}\times\mathcal{M}} g_m(\mathbf{x}) g_n(\mathbf{x})\mu(d\mathbf{x}) \tag{20}$$

By letting $\mathbf{x} = [f\, a\, \phi]^T$ and

$$g_m(\mathbf{x}) = \frac{a}{\sqrt{\lambda_0/2}}\cos(2\pi f m + \phi)$$

and also

$$\mu(d\mathbf{x}) = \lambda_0 p(f) p_A(a) p_\Phi(\phi) d\phi\, da\, df$$

we can find the moments of $X[n]$.

The mean is derived first by using (19).

$$E[X[n]] = \int_0^{1/2}\int_0^{\infty}\int_0^{2\pi} \frac{a}{\sqrt{\lambda_0/2}}\cos(2\pi f n + \phi)\lambda_0 p(f) p_A(a) p_\Phi(\phi) d\phi\, da\, df = 0$$

due to the integration over ϕ.

Next the autocorrelation sequence and the PSD are found.

$$\begin{aligned} E[X[m]X[n]] &= \int_0^{1/2}\int_0^{\infty}\int_0^{2\pi} \frac{a^2}{\lambda_0/2}\cos(2\pi f m + \phi)\cos(2\pi f n + \phi)\lambda_0 p(f) p_A(a) p_\Phi(\phi) d\phi\, da\, df \\ &= 2E[A^2]\int_0^{1/2}\int_0^{2\pi} \frac{1}{2}\cos[2\pi f(m-n)] + \frac{1}{2}\cos[2\pi f(m+n) + 2\phi]\frac{1}{2\pi} d\phi\, p(f) df \\ &= 2E[A^2]\int_0^{1/2} \frac{1}{2}\cos[2\pi f(m-n)] p(f) df \\ &= r_X[m-n] \end{aligned}$$

so that the autocorrelation sequence is

$$r_X[k] = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{E[A^2]p(|f|)}{2}\cos(2\pi f k) df$$

and the PSD is seen to be

$$P_X(f) = \frac{E[A^2]p(|f|)}{2} \qquad -1/2 \le f \le 1/2.$$

B Derivation of Ergodicity of Sample Autocorrelation

From [3] there are two conditions that are necessary and sufficient for ergodicity. They are

1. The fourth moment $E[X[n_0]X[k+n_0]X[j+n_0]X[j+k+n_0]]$ should not depend on n_0, which is a form of stationarity for this moment.

2. If the sample autocorrelation is given by

$$\hat{r}_X[k] = \frac{1}{M+1}\sum_{j=0}^{M+1} X[j]X[j+k]$$

then we require

$$\lim_{M\to\infty}\frac{1}{M+1}\sum_{j=0}^{M+1} E[X[0]]E[X[k]]E[X[j]]E[X[j+k]] = r_X^2[k]$$

and is equivalent to requiring the variance of $\hat{r}_X[k]$ to go to zero as the data record length M goes to ∞. We first verify the stationarity of the fourth-order moment.

To do so we let

$$g_n(\mathbf{x}) = X[n] = \frac{a}{\sqrt{\lambda_0/2}}\cos(2\pi f n + \phi)$$

and use the fourth-order moment results derived in Appendix C. We have

$$\begin{aligned}
E[X[n_0]X[k+n_0]X[j+n_0]X[j+k+n_0]] &= \int_{\mathcal{S}\times\mathcal{M}} g_{n_0}(\mathbf{x})g_{k+n_0}(\mathbf{x})g_{j+n_0}(\mathbf{x})g_{j+k+n_0}(\mathbf{x})\mu(d\mathbf{x})\\
&+ \int_{\mathcal{S}\times\mathcal{M}} g_{n_0}(\mathbf{x})g_{k+n_0}(\mathbf{x})\mu(d\mathbf{x})\int_{\mathcal{S}\times\mathcal{M}} g_{j+n_0}(\mathbf{x})g_{j+k+n_0}(\mathbf{x})\mu(d\mathbf{x})\\
&+ \int_{\mathcal{S}\times\mathcal{M}} g_{n_0}(\mathbf{x})g_{j+n_0}(\mathbf{x})\mu(d\mathbf{x})\int_{\mathcal{S}\times\mathcal{M}} g_{k+n_0}(\mathbf{x})g_{j+k+n_0}(\mathbf{x})\mu(d\mathbf{x})\\
&+ \int_{\mathcal{S}\times\mathcal{M}} g_{n_0}(\mathbf{x})g_{j+k+n_0}(\mathbf{x})\mu(d\mathbf{x})\int_{\mathcal{S}\times\mathcal{M}} g_{k+n_0}(\mathbf{x})g_{j+n_0}(\mathbf{x})\mu(d\mathbf{x})
\end{aligned}$$

and from Appendix A

$$\int_{\mathcal{S}\times\mathcal{M}} g_m(\mathbf{x})g_n(\mathbf{x})\mu(\mathbf{x}) = r_X[m-n].$$

Thus, the last three terms are

$$r_X^2[k] + r_X^2[j] + r_X[j+k]r_X[j-k]$$

and clearly do not depend on n_0. Considering the first term, which we denote by I, we have

$$\begin{aligned}
I &= \int_0^{1/2}\int_0^{\infty}\int_0^{2\pi}\frac{1}{(\lambda_0/2)^2}a^4\cos[2\pi f n_0+\phi]\cos[2\pi f(k+n_0)+\phi]\\
&\cdot\ \cos[2\pi f(j+n_0)+\phi]\cos[2\pi f(j+k+n_0)+\phi]\lambda_0 p_\Phi(\phi)p_A(a)p(f)\,d\phi\,da\,df\\
&= \frac{4E[A^4]}{\lambda_0}\int_0^{1/2}\int_0^{2\pi}[\cos[2\pi f n_0+\phi]\cos[2\pi f(k+n_0)+\phi]\\
&\cdot\ \cos[2\pi f(j+n_0)+\phi]\cos[2\pi f(j+k+n_0)+\phi]]\,\frac{1}{2\pi}p(f)\,d\phi\,df
\end{aligned}$$

To evaluate the integral over ϕ we let $z_i = \exp(j\theta_i)$, $i = 1, 2, 3, 4$ with

$$\begin{aligned}
\theta_1 &= 2\pi f n_0 + \phi \\
\theta_2 &= 2\pi f(k + n_0) + \phi \\
\theta_3 &= 2\pi f(j + n_0) + \phi \\
\theta_4 &= 2\pi f(j + k + n_0) + \phi
\end{aligned}$$

so that the fourth-order product of cosines in brackets becomes

$$\frac{1}{16}\prod_{i=1}^{4}(z_i + z_i^*).$$

When multiplied out, only the product terms that have two unconjugated z_i's and two conjugated z_i's so that the term does not depend on ϕ will produce a nonzero contribution to the integral. It can be shown that this results in the terms

$$\frac{1}{8}\left[\cos(\theta_1 + \theta_2 - \theta_3 - \theta_4) + \cos(\theta_1 - \theta_2 + \theta_3 - \theta_4) + \cos(\theta_1 - \theta_2 - \theta_3 + \theta_4)\right]$$

which is

$$\frac{1}{8}\left[\cos(-2\pi fj - 2\pi fj) + \cos(-2\pi fk - 2\pi fk) + \cos(-2\pi fk + 2\pi fk)\right] = \frac{1}{8}\left[\cos(4\pi fj) + \cos(4\pi fk) + 1\right].$$

At this point we see that the fourth-order moment does not depend on n_0 and hence the first condition for ergodicity is satisfied. Continuing on we compute the fourth-order moment $E[X[0]X[k]X[j][j+k]]$, which is just the previous expression with $n_0 = 0$.

We now continue the evaluation of I.

$$\begin{aligned}
I &= \frac{4E[A^4]}{\lambda_0}\int_0^{1/2}\int_0^{2\pi}\left[\cos[2\pi f n_0 + \phi]\cos[2\pi f(k+n_0) + \phi]\right. \\
&\cdot \left.\cos[2\pi f(j+n_0) + \phi]\cos[2\pi f(j+k+n_0) + \phi]\right]\frac{1}{2\pi}p(f)d\phi df \\
&= \frac{E[A^4]}{2\lambda_0}\int_0^{1/2}\int_0^{2\pi}\left[\cos(4\pi fj) + \cos(4\pi fk) + 1\right]\frac{1}{2\pi}p(f)d\phi df \\
&= \frac{E[A^4]}{2\lambda_0}\int_0^{1/2}\left[\cos(4\pi fj) + \cos(4\pi fk) + 1\right]p(f)df \\
&= \frac{E[A^4]}{2\lambda_0 E[A^2]}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left[\cos(2\pi fj) + \cos(4\pi fk) + 1\right]\frac{E[A^2]p(|f|)}{2}df \\
&= \frac{E[A^4]}{2\lambda_0 E[A^2]}[r_X[2j] + r_X[2k] + r_X[0]]
\end{aligned}$$

so that

$$E[X[0]X[k]X[j]X[j+k]] = \frac{E[A^4]}{2\lambda_0 E[A^2]}[r_X[2j] + r_X[2k] + r_X[0]] + r_X^2[k] + r_X^2[j] + r_X[j+k]r_X[j-k].$$

Thus,

$$\begin{aligned}
\lim_{M\to\infty} \frac{1}{M+1}\sum_{j=0}^{M} E[X[0]X[k]X[j]X[j+k]] &= \lim_{M\to\infty} \frac{1}{M+1}\sum_{j=0}^{M} \frac{E[A^4]}{2\lambda_0 E[A^2]}(r_X[2j]+r_X[2k]+r_X[0]) \\
&+ \lim_{M\to\infty} \frac{1}{M+1}\sum_{j=0}^{M}(r_X^2[j]+r_X[j+k]r_X[j-k])+r_X^2[k]
\end{aligned}$$

and assuming that $\sum_{j=0}^{M}|r_X[j]|<\infty$ and $\sum_{j=0}^{M}|r_X^2[j]+r_X[j+k]r_X[j-k]|<\infty$, which will be true for an absolutely continuous spectral measure, we see that

$$\lim_{M\to\infty} \frac{1}{M+1}\sum_{j=0}^{M} E[X[0]X[k]X[j]X[j+k]] = \frac{E[A^4]}{2\lambda_0 E[A^2]}(r_X[2k]+r_X[0])+r_X^2[k]$$

which will only approach $r_X^2[k]$ as $\lambda_0\to\infty$.

C Derivation of Joint Characteristic Function

The principal approach to determining properties of a Poisson process is the characteristic function and Campbell's theorem [13]. The general fourth-order moments necessary do not appear in the literature and so this appendix fills that gap. In the process we will also derive the lower-order moments, some of which are in [13], as well as many other references. We use a general procedure to allow the application to any Poisson process.

The joint characteristic function of $\mathbf{Z}=[Z_1\, Z_2 \ldots Z_p]^T$ as given by (18) can be shown to be

$$\psi_{\mathbf{z}}(\boldsymbol{\omega}) = E[\exp(j\boldsymbol{\omega}^T\mathbf{Z})] = \exp\left[\int_{\mathcal{S}\times\mathcal{M}}\left(\exp[j\boldsymbol{\omega}^T\mathbf{g}(\mathbf{x})]-1\right)\mu(d\mathbf{x})\right]$$

where $\boldsymbol{\omega}=[\omega_1\,\omega_2\ldots\omega_p]^T$, $\mathbf{g}(\mathbf{x})=[g_1(\mathbf{x})\,g_2(\mathbf{x})\ldots g_p(\mathbf{x})]^T$, and $\mu(\mathcal{A})$ is the mean measure of the set $\mathcal{A}$. It is asumed that the integral exists, which is assured if $\mu(\mathcal{S}\times\mathcal{M})<\infty$. It can be shown by Campbell's theorem that

$$E[g_i(\mathbf{X})] = \int_{\mathcal{S}\times\mathcal{M}} g_i(\mathbf{x})\mu(d\mathbf{x})$$

and assuming this equals zero, we have that

$$\begin{aligned}
\psi_{\mathbf{z}}(\boldsymbol{\omega}) &= \exp\left[\int_{\mathcal{S}\times\mathcal{M}}\left(\exp[j\boldsymbol{\omega}^T\mathbf{g}(\mathbf{x})]-j\boldsymbol{\omega}^T\mathbf{g}(\mathbf{x})-1\right)\mu(d\mathbf{x})\right] \\
&= \exp\left[\int_{\mathcal{S}\times\mathcal{M}}\sum_{k=2}^{\infty}\frac{(j\boldsymbol{\omega}^T\mathbf{g}(\mathbf{x}))^k}{k!}\mu(d\mathbf{x})\right] \\
&= \exp\left[\sum_{k=2}^{\infty}\int_{\mathcal{S}\times\mathcal{M}}\frac{(j\boldsymbol{\omega}^T\mathbf{g}(\mathbf{x}))^k}{k!}\mu(d\mathbf{x})\right]
\end{aligned}$$

with the last step justified via the Beppo-Levi theorem and the assumption that

$$\sum_{k=2}^{\infty}\int_{\mathcal{S}\times\mathcal{M}}\left|\frac{(j\boldsymbol{\omega}^T\mathbf{g}(\mathbf{x}))^k}{k!}\right|\mu(d\mathbf{x})<\infty.$$

Next to differentiate the characteristic function it is convenient to let

$$G(\boldsymbol{\omega}, \nu, h) = \sum_{k=\nu}^{\infty} \int_{\mathcal{S}\times\mathcal{M}} \frac{(\sum_{i=1}^{p} j\omega_i g_i(\mathbf{x}))^k}{k!} h(\mathbf{x})\mu(d\mathbf{x})$$

where $\nu \geq 0$ so that we have

$$\psi_{\mathbf{z}}(\boldsymbol{\omega}) = \exp(G(\boldsymbol{\omega}, 2, e))$$

and $e(\mathbf{x}) = 1$. Note that

$$\frac{\partial G(\boldsymbol{\omega}, \nu, e)}{\partial \omega_m} = \begin{cases} G(\boldsymbol{\omega}, \nu - 1, jg_m) & \nu \geq 1 \\ G(\boldsymbol{\omega}, 0, jg_m) & \nu < 1 \end{cases}$$

so that if the second argument of G is less than zero, it should be set to zero. Similarly, we have

$$\begin{aligned} \frac{\partial^2 G}{\partial \omega_m \partial \omega_n} &= G(\boldsymbol{\omega}, \nu - 2, j^2 g_m g_n) \\ \frac{\partial^3 G}{\partial \omega_m \partial \omega_n \partial \omega_r} &= G(\boldsymbol{\omega}, \nu - 3, j^3 g_m g_n g_r) \\ = \frac{\partial^4 G}{\partial \omega_m \partial \omega_n \partial \omega_r \partial \omega_s} &= G(\boldsymbol{\omega}, \nu - 4, j^4 g_m g_n g_r g_s). \end{aligned}$$

Also we make use of the relationship

$$G(\boldsymbol{\omega}, \nu, h)|_{\boldsymbol{\omega}=\mathbf{0}} = \begin{cases} \int_{\mathcal{S}\times\mathcal{M}} h(\mathbf{x})\mu(d\mathbf{x}) & \nu = 0 \\ 0 & \nu \geq 1 \end{cases}$$

As a result we obtain the moments as follows. They are

$$\begin{aligned} E[Z_m] &= \frac{1}{j} \left. \frac{\partial \psi_{\mathbf{z}}}{\partial \omega_m} \right|_{\boldsymbol{\omega}=\mathbf{0}} \\ &= \frac{1}{j} \psi_{\mathbf{z}}(\mathbf{0}) G(\mathbf{0}, 1, jg_m) = 0. \end{aligned}$$

$$E[Z_m Z_n] = \frac{1}{j^2} \left. \frac{\partial^2 \psi_{\mathbf{z}}}{\partial \omega_m \partial \omega_n} \right|_{\boldsymbol{\omega}=\mathbf{0}}$$

and

$$\begin{aligned} \frac{\partial^2 \psi_{\mathbf{z}}}{\partial \omega_m \partial \omega_n} &= \frac{\partial}{\partial \omega_m} [\psi_{\mathbf{z}}(\boldsymbol{\omega}) G(\boldsymbol{\omega}, 1, jg_m)] \\ &= \psi_{\mathbf{z}}(\boldsymbol{\omega}) G(\boldsymbol{\omega}, 0, j^2 g_m g_n) + \psi_{\mathbf{z}}(\boldsymbol{\omega}) G(\boldsymbol{\omega}, 1, jg_n) G(\boldsymbol{\omega}, 1, jg_m). \end{aligned}$$

Evaluating this at $\boldsymbol{\omega} = \mathbf{0}$ produces $G(\mathbf{0}, 0, j^2 g_m g_n)$ or finally

$$E[Z_m Z_n] = \int_{\mathcal{S}\times\mathcal{M}} g_m(\mathbf{x}) g_n(\mathbf{x}) \mu(d\mathbf{x}).$$

The first and second moment are just Campbell's theorem. Next

$$E[Z_m Z_n Z_r] = \frac{1}{j^3} \left. \frac{\partial^3 \psi_{\mathbf{z}}}{\partial \omega_m \partial \omega_n \partial \omega_r} \right|_{\boldsymbol{\omega}=\mathbf{0}}$$

and

$$
\begin{aligned}
\frac{\partial^3 \psi_{\mathbf{z}}}{\partial \omega_m \partial \omega_n \partial \omega_r} &= \frac{\partial}{\partial \omega_r}\left[\psi_{\mathbf{z}}(\boldsymbol{\omega})G(\boldsymbol{\omega},0,j^2 g_m g_n) + \psi_{\mathbf{z}}(\boldsymbol{\omega})G(\boldsymbol{\omega},1,jg_n)G(\boldsymbol{\omega},1,jg_m)\right] \\
&= \psi_{\mathbf{z}}(\boldsymbol{\omega})G(\boldsymbol{\omega},0,j^3 g_m g_n g_r) + \psi_{\mathbf{z}}(\boldsymbol{\omega})G(\boldsymbol{\omega},1,jg_r)G(\boldsymbol{\omega},0,j^2 g_m g_n) \\
&+ \psi_{\mathbf{z}}(\boldsymbol{\omega})\left[G(\boldsymbol{\omega},0,j^2 g_n g_r)G(\boldsymbol{\omega},1,jg_m) + G(\boldsymbol{\omega},1,jg_n)G(\boldsymbol{\omega},0,j^2 g_m g_r)\right] \\
&+ \psi_{\mathbf{z}}(\boldsymbol{\omega})G(\boldsymbol{\omega},1,jg_r)G(\boldsymbol{\omega},1,jg_n)G(\boldsymbol{\omega},1,jg_m).
\end{aligned}
$$

Finally, we have

$$
E[Z_m Z_n Z_r] = \int_{\mathcal{S}\times\mathcal{M}} g_m(\mathbf{x})g_n(\mathbf{x})g_r(\mathbf{x})\mu(d\mathbf{x}).
$$

The fourth-order moment is found similarly as

$$
E[Z_m Z_n Z_r Z_s] = \frac{1}{j^4}\left.\frac{\partial^4 \psi_{\mathbf{z}}}{\partial \omega_m \partial \omega_n \partial \omega_r \partial \omega_s}\right|_{\boldsymbol{\omega}=\mathbf{0}}.
$$

Note that only the third derivative terms above that have a factor of $G(\cdot,0,\cdot)$ after being differentiated will be nonzero when $\boldsymbol{\omega}=\mathbf{0}$. This produces the fourth-order derivative evaluated at $\boldsymbol{\omega}=\mathbf{0}$ of

$$
\begin{aligned}
&\psi_{\mathbf{z}}(\boldsymbol{\omega})G(\boldsymbol{\omega},0,j^4 g_m g_n g_r g_s) + \psi_{\mathbf{z}}(\boldsymbol{\omega})G(\boldsymbol{\omega},0,j^2 g_r g_s)G(\boldsymbol{\omega},0,j^2 g_m g_n) \\
+\ &\psi_{\mathbf{z}}(\boldsymbol{\omega})G(\boldsymbol{\omega},0,j^2 g_n g_r)G(\boldsymbol{\omega},0,j^2 g_m g_s) + \psi_{\mathbf{z}}(\boldsymbol{\omega})G(\boldsymbol{\omega},0,j^2 g_n g_s)G(\boldsymbol{\omega},0,j^2 g_m g_r)
\end{aligned}
$$

and finally we have

$$
\begin{aligned}
E[Z_m Z_n Z_r Z_s] &= \int_{\mathcal{S}\times\mathcal{M}} g_m(\mathbf{x})g_n(\mathbf{x})g_r(\mathbf{x})g_s(\mathbf{x})\mu(d\mathbf{x}) + \int_{\mathcal{S}\times\mathcal{M}} g_r(\mathbf{x})g_s(\mathbf{x})\mu(d\mathbf{x}) \int_{\mathcal{S}\times\mathcal{M}} g_m(\mathbf{x})g_n(\mathbf{x})\mu(d\mathbf{x}) \\
&+ \int_{\mathcal{S}\times\mathcal{M}} g_n(\mathbf{x})g_r(\mathbf{x})\mu(d\mathbf{x}) \int_{\mathcal{S}\times\mathcal{M}} g_m(\mathbf{x})g_s(\mathbf{x})\mu(d\mathbf{x}) \\
&+ \int_{\mathcal{S}\times\mathcal{M}} g_n(\mathbf{x})g_s(\mathbf{x})\mu(d\mathbf{x}) \int_{\mathcal{S}\times\mathcal{M}} g_m(\mathbf{x})g_r(\mathbf{x})\mu(d\mathbf{x}).
\end{aligned}
$$

D Derivation of First-Order PDF

If

$$
X[n] = \sum_{\mathbf{x}\in\Pi^*} g_n(\mathbf{X})
$$

where

$$
g_n(\mathbf{x}) = \frac{a}{\sqrt{\lambda_0/2}}\cos(2\pi f n + \phi)
$$

then the characteristic function can be shown to be [13]

$$
\begin{aligned}
\psi_{X[n]}(\omega) &= \exp\left[\int_{\mathcal{S}\times\mathcal{M}} \left(\exp(j\omega g_n(\mathbf{x})) - 1\right)\mu(d\mathbf{x})\right] \\
&= \exp\left[\int_0^{1/2}\int_0^{\infty}\int_0^{2\pi}\left(\exp\left(j\omega\frac{a}{\sqrt{\lambda_0/2}}\cos(2\pi f n + \phi)\right) - 1\right)\lambda_0 p(f)p_A(a)p_\Phi(\phi)d\phi\, da\, df\right].
\end{aligned}
$$

Since Φ is uniform over the interval $[0, 2\pi)$, by integrating over ϕ we obtain

$$\psi_{X[n]}(\omega) = \exp\left[\int_0^{1/2}\int_0^{\infty}\left(J_0\left(\omega\frac{a}{\sqrt{\lambda_0/2}}\right) - 1\right)\lambda_0 p(f)p_A(a)da\,df\right]$$

and next integrating over f produces

$$\psi_{X[n]}(\omega) = \exp\left[\lambda_0\int_0^{\infty}\left(J_0\left(\omega\frac{a}{\sqrt{\lambda_0/2}}\right) - 1\right)p_A(a)da\right]$$

or the characteristic function is that of a compound Poisson random variable

$$\psi_{X[n]}(\omega) = \exp\left[\lambda_0(\psi_U(\omega) - 1)\right]$$

where

$$\psi_U(\omega) = \int_0^{\infty} J_0\left(\omega\frac{a}{\sqrt{\lambda_0/2}}\right)p_A(a)da.$$

E Derivation of Convergence to Gaussian Random Process

Consider an arbitrary number of samples K at arbitrary times $\{n_1, n_2, \ldots, n_K\}$. The characteristic function of $\mathbf{Z} = [X[n_1]\,X[n_2]\ldots X[n_K]]^T$ was shown in Appendix C to be given by

$$\psi_{\mathbf{z}}(\boldsymbol{\omega}) = \exp\left[\sum_{k=2}^{\infty}\int_{\mathcal{S}\times\mathcal{M}}\frac{(j\boldsymbol{\omega}^T\mathbf{g}(\mathbf{x}))^k}{k!}\mu(d\mathbf{x})\right]$$

where $\mathbf{x} = [f\,a\,\phi]^T$ and

$$x[n_i] = g_i(\mathbf{x}) = \frac{1}{\sqrt{\lambda_0/2}}a\cos(2\pi f n_i + \phi).$$

Thus, we have

$$\begin{aligned}
\ln\psi_{\mathbf{z}}(\boldsymbol{\omega}) &= \sum_{k=2}^{\infty}\int_0^{1/2}\int_0^{\infty}\int_0^{2\pi}\frac{1}{k!}\left(j\sum_{i=1}^{K}\omega_i\frac{1}{\sqrt{\lambda_0/2}}a\cos(2\pi f n_i + \phi)\right)^k\lambda_0 p(f)p_A(a)p_\Phi(\phi)d\phi\,da\,df \\
&= \sum_{k=2}^{\infty}\int_0^{1/2}\int_0^{\infty}\int_0^{2\pi}\frac{1}{\lambda_0^{k/2-1}k!}\left(j\sqrt{2}\sum_{i=1}^{K}\omega_i a\cos(2\pi f n_i + \phi)\right)^k p(f)p_A(a)p_\Phi(\phi)d\phi\,da\,df \\
&= \int_0^{1/2}\int_0^{\infty}\int_0^{2\pi}\frac{1}{2!}\left(j\sqrt{2}\sum_{i=1}^{K}\omega_i a\cos(2\pi f n_i + \phi)\right)^2 p(f)p_A(a)p_\Phi(\phi)d\phi\,da\,df + O(1/\sqrt{\lambda_0})
\end{aligned}$$

and as $\lambda_0 \to \infty$

$$\begin{aligned}
\ln\psi_{\mathbf{z}}(\boldsymbol{\omega}) &\to -E[A^2]\int_0^{1/2}\int_0^{2\pi}\left(\sum_{i=1}^{K}\omega_i\cos(2\pi f n_i + \phi)\right)^2 p(f)p_\Phi(\phi)d\phi\,df \\
&= -\frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{K}\omega_i\omega_j[\mathbf{A}]_{ij}
\end{aligned}$$

where

$$
\begin{aligned}
[\mathbf{A}]_{ij} &= 2E[A^2]\int_0^{1/2}\int_0^{2\pi} \cos(2\pi f n_i + \phi)\cos(2\pi f n_j + \phi)p(f)p_\Phi(\phi)d\phi\, df \\
&= E[A^2]\int_0^{1/2} \cos(2\pi f(n_i - n_j))p(f)df.
\end{aligned}
$$

Therefore, we have that

$$
\begin{aligned}
[\mathbf{A}]_{ij} &= \int_0^{1/2} \cos(2\pi f(n_i - n_j))p(f)df \\
&= \int_0^{1/2} \cos(2\pi f(n_i - n_j))E[A^2]p(f)df \\
&= \int_{-1/2}^{1/2} \cos(2\pi f(n_i - n_j))\left(\frac{E[A^2]}{2}p(|f|)\right)df \\
&= r_X[n_i - n_j].
\end{aligned}
$$

Thus,

$$
\psi_{\mathbf{z}}(\boldsymbol{\omega}) \rightarrow \exp\left(-\frac{1}{2}\boldsymbol{\omega}^T\mathbf{C}\boldsymbol{\omega}\right)
$$

where

$$
[\mathbf{C}]_{ij} = r_X[n_i - n_j]
$$

and $\mathbf{C}$ is recognized as the covariance matrix, from which we can conclude that the random process approaches a Gaussian random process as $\lambda_0 \rightarrow \infty$.