Generalizing Stochastic Resonance by the Transformation Method

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Abstract

For a decision rule whose decision region is suboptimal we show how to transform the decision statistic to recover optimal performance. The procedure simply amounts to transforming the decision statistic to yield a combined statistic/decision function which is optimal. The approach may be thought of as a generalization of the stochastic resonance phenomenon, which employs a random linear transformation, and hence should be widely applicable to practical problems.

1 Introduction

The phenomenon of stochastic resonance [1] is that by adding noise to an observed test statistic one may sometimes improve the detection performance of a fixed but *suboptimal* detector [2]. A detailed analysis of this phenomenon from a statistical detection theory viewpoint is contained in [3]. For example, assume that we decide a signal is present if a test statistic exceeds a threshold or if $X > \gamma$. If we add a noise sample to X to form $Y = X + U$ and then compare it to a threshold, we will decide a signal is present if $X + U > \gamma$. This is equivalent to deciding a signal is present if $X > \gamma - U$, which is seen to be a

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transformation of the decision region, although a stochastic one. One wonders if there are more general methods of transforming the decision region that will improve the detectability. We show in the next section how a suboptimal decision region can be effectively transformed into an optimal one by transforming X using a *nonlinear* transformation. The method is completely general in that any suboptimal decision rule may be transformed into an optimal one, under some mild conditions.

2 Mathematical Description of Transformation Method

A mathematical justification of the approach is given in this section with an example in the next section. The reader may wish to skip this section and proceed directly to the example, as the latter more clearly indicates the procedure.

We assume that the problem is to decide between two hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$ based on an observed scalar test statistic x. The two hypotheses are assumed to be random events with prior probabilities of π_0 and π_1. This test statistic is a function of the original data. A future paper will address the extension to the case when the original data is accessible. Based on the observed data sample x a decision rule has been implemented as follows:

$$\begin{aligned} \phi(x) &= 1 \quad \text{decide } \mathcal{H}_1 \\ \phi(x) &= 0 \quad \text{decide } \mathcal{H}_0. \end{aligned}$$

This decision rule is assumed to be *suboptimal.* Denoting the probability density functions (PDFs) as $p_0^X(x)$ and $p_1^X(x)$ under $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively, the probability of a correct decision is

$$\begin{aligned} P_c &= \pi_0 \int_{-\infty}^{\infty} (1-\phi(x)) p_0^X(x) dx + \pi_1 \int_{-\infty}^{\infty} \phi(x) p_1^X(x) dx \\ &= \pi_0 + \int_{-\infty}^{\infty} \phi(x)(\pi_1 p_1^X(x) - \pi_0 p_1^X(x))) dx \end{aligned}$$

Note that unless $\phi(x) = 1$ for all x such that $\pi_1 p_1^X(x) - \pi_0 p_0^X(x) > 0$ and zero otherwise (which is the optimal decision rule) this probability will not be maximized.

Now consider that we transform the test statistic as $y = g(x)$ using some function g. The function is assumed to be piece-wise monotonic so that over each interval of a finite number of disjoint intervals either $g'(x) \geq 0$ or $g'(x) < 0$ (the prime denotes differentiation). The transformed test statistic y is then input to the decision rule $\phi(\cdot)$ to yield $\phi(y)$. This is inaccordance with the assumption that the decision rule is fixed and so cannot be changed, only the test statistic can be modified. It is assumed that the cumulative distribution function of Y is continuous, i.e., that Y is a continuous random variable. We will see next that this transformation of the test statistic is mathematically equivalent to modifying the decision rule.

To do so note that P_c, which is now based on y, is

$$P_c = \pi_0 + \int_{-\infty}^{\infty} \phi(y)(\pi_1 p_1^Y(y) - \pi_0 p_1^Y(y))dy.$$

We now utilize the piece-wise monotonic assumption of $g(\cdot)$ to write a subset of the real line as $S = \cup_{i=1}^N I_i = \cup_{i=1}^N (a_i, b_i)$, where $a_1 < b_1 < a_2 < b_2 \ldots < a_N < b_N$ and the intervals are open. The function $g(\cdot)$ is monotonic over each interval I_i. The omission of a finite number of points from the real line R and subsequently from any integral will not affect the results as long as the PDFs do not contain any impulses at these points (or the cumulative distribution function is continuous over all of R). Now we have that by defining

$$J_i = \begin{cases} (a_i, b_i) & \text{if } g'(I_i) \geq 0 \\ (b_i, a_i) & \text{if } g'(I_i) < 0 \end{cases}$$

and using a change of variables from y to $g(x)$

$$\begin{aligned} P_c &= \pi_0 + \int \phi(g(x))(\pi_1 p_1^Y(g(x)) - \pi_0 p_0^Y(g(x)))g'(x)dx \\ &= \pi_0 + \sum_{i=1}^N \int_{J_i} \phi(g(x))(\pi_1 p_1^Y(g(x)) - \pi_0 p_0^Y(g(x)))g'(x)dx. \end{aligned}$$

Note that for the intervals for which $g'(x) < 0$, we have $J_i = (b_i, a_i)$. Absorbing the negative sign into $g'(x)$ for the monotonically decreasing function intervals yields

$$P_c = \pi_0 + \sum_{i=1}^N \int_{I_i} \phi(g(x))(\pi_1 p_1^Y(g(x)) - \pi_0 p_0^Y(g(x))) \left|g'(x)\right| dx.$$

Next we recognize that $p_1^Y(g(x))\,|g'(x)| = p_1^X(x)$ and $p_0^Y(g(x))\,|g'(x)| = p_0^X(x)$ so that we have finally

$$\begin{aligned} P_c &= \pi_0 + \sum_{i=1}^N \int_{I_i} \phi(g(x))(\pi_1 p_1^X(x) - \pi_0 p_0^X(x))dx \\ &= \pi_0 + \int_{-\infty}^{\infty} \phi(g(x))(\pi_1 p_1^X(x) - \pi_0 p_0^X(x))dx. \end{aligned} \qquad (1)$$

We now see that the probability of a correct decision is based on $\phi(g(x))$. In effect by *transforming the decision test statistic x to $g(x)$ we have been able to effectively modify the decision region.* It is clear from (1) that for optimal performance we must have

$$\phi^*(x) = \phi(g(x)) = \begin{cases} 1 & \text{if } \pi_1 p_1^X(x) > \pi_0 p_0^X(x) \\ 0 & \text{otherwise} \end{cases}$$

where $\phi^*(x)$ denotes the optimal decision rule based on x. In composite function notation, we require for optimality that

$$\phi \circ g(x) = \phi(g(x)) = \phi^*(x). \qquad (2)$$

We need only determine the function $g(\cdot)$. We provide an example in the next section.

3 An Example

We consider a very simple example, for which the solution is obvious. The example is that of deciding between $x \sim \mathcal{N}(0,1)$ under $\mathcal{H}_0$ and $x \sim \mathcal{N}(1,1)$ under $\mathcal{H}_1$, where $\mathcal{N}(\mu,\sigma^2)$ denotes a Gaussian PDF with mean μ and variance σ^2. The prior probabilities are $\pi_0 = \pi_1 = 1/2$. We assume that the *suboptimal* decision rule is to decide $\mathcal{H}_1$ if $x \geq 0$ and decide $\mathcal{H}_0$ if $x < 0$. The optimal decision rule for this problem is to decide $\mathcal{H}_1$ if $x \geq 1/2$ and to decide $\mathcal{H}_0$ if $x < 1/2$. This is just the maximum likelihood decision rule [4]. The suboptimal decision rule produces the correct decision for all x not in the interval $[0, 1/2)$ as shown in Figure 1. It is clear now that to modify the suboptimal decision rule to make it optimal we need only

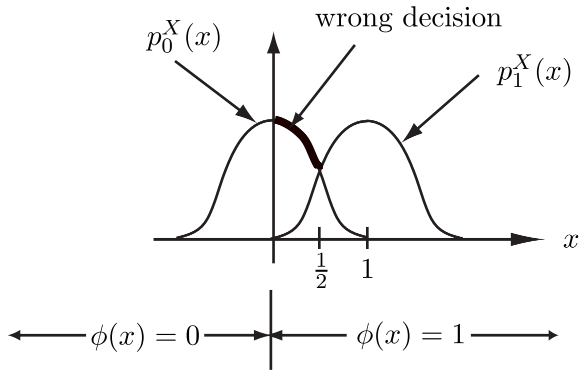

Figure 1: Example of suboptimal decision regions.

map the values of x in the interval $[0, 1/2)$ into any other interval for which the suboptimal decision rule will produce a zero at its output. For example, we could use

$$g(x) = \begin{cases} x & \text{for } x \geq 1/2 \text{ and } x < 0 \\ -x & \text{for } 0 \leq x < 1/2 \end{cases}$$

as shown in Figure 2. Note that the effect of the transformation is to do nothing ($g(x) = x$) if the test statistic value will produce the correct decision. However, in the interval $[0, 1/2)$ the decision is incorrect. To convert it to a correct decision we simply negate the value of the test statistic as $g(x) = -x$. Then, the values $0 \leq x < 1/2$ become negative and are decided to correspond to $\mathcal{H}_0$ in accordance with the suboptimal decision rule. Finally, it should be observed that the function chosen is piece-wise monotonic (as well as discontinuous).

References

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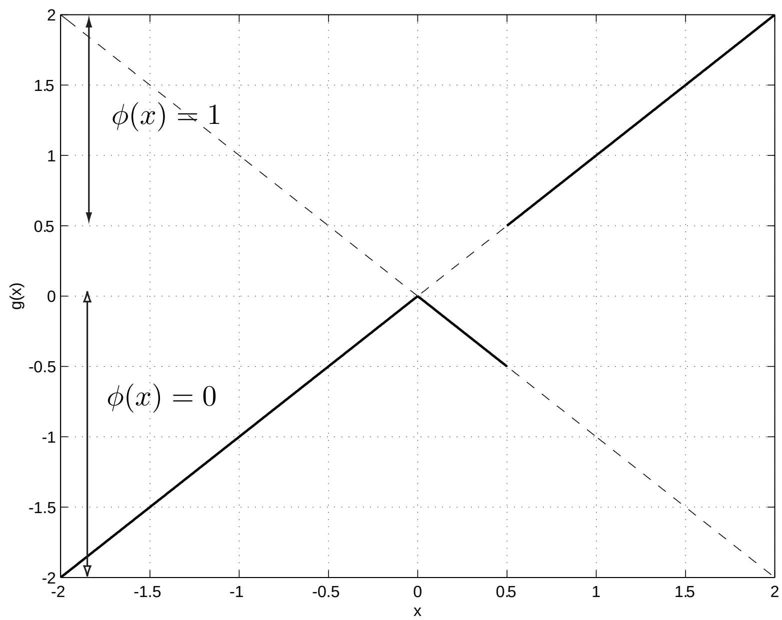

Figure 2: Transforming function - one of many possibilities.

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