Scattering of Plane Electromagnetic Waves at the Junction Formed by a PEC half-plane and a half-plane with Anisotropic Conductivity

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Abstract

In this study, scattering of plane electromagnetic waves at the junction formed by a PEC half-plane and a half-plane with anisotropic conductivity is investigated. By using Fourier Transform technique the problem is formulated into a matrix Wiener-Hopf system and an exact closed-form solution is obtained for the most general case by factorizing a $2 \times 2$ Wiener-Hopf matrix. Also, four different special cases are examined which in all, by using Fourier Transform technique the problem is formulated into a pair of simultaneous Wiener-Hopf equations which are decoupled via a polynomial transformation and solved through the standard procedure. The diffracted field is expressed in a form suitable for GTD applications and the effects of the resistivities of the anisotropic half-plane on the diffraction coefficient are also investigated.

I. Introduction

As is known, the scattering from any body is a function of both its geometrical and material properties. In recent years there has been a renewed interest in investigating the influence of material properties on edge diffraction. In particular, the edge diffraction by a half plane of finite, isotropic conductivity has been studied by several authors [1, 2]. In these works, the electromagnetic property of imperfectly conducting surface is specified by its scalar resistivity $R = (1/\sigma t)$ with $\sigma$ being the conductivity, and $t$ is the thickness which is assumed to be small compared with the wavelength. In the most general case, as noted by Senior[3], the surface resistivity may be anisotropic supporting electric current sheets in directions parallel to both axes of the plane. For such a plane located at $y = 0$, a constant resistivity tensor can be written in the following dyadic form:

$$\mathbf{R} = R_1 \hat{\mathbf{x}} \hat{\mathbf{x}} + R_2 \hat{\mathbf{z}} \hat{\mathbf{z}}$$  \hspace{1cm} (1)

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ denote the unit vectors in Cartesian coordinate system. Here, $R_1$ and $R_2$ represent

$$R_1 = (1/\sigma_x t) \hspace{1cm} \text{and} \hspace{1cm} R_2 = (1/\sigma_z t) \hspace{1cm} (2)$$

with $\sigma_x$ and $\sigma_z$ being the conductivities in the $x$ and $z$ directions, respectively. In this work, scattering of plane electromagnetic waves at the junction formed by a PEC half-plane and a half-plane with anisotropic conductivity is considered for the oblique incidence case. The PEC half-plane is located at $y = 0$, $x < 0$ and at $y = 0$, $x > 0$ another half-plane is located with anisotropic resistivity where the half-plane is of finite resistivities $R_2$ and $R_1$ in the directions $z$ and $x$, respectively (see Fig.-1). The structure is simulated by standard resistive boundary conditions and the problem is formulated by Fourier transform technique. The formal solution is derived for the diffraction problem by employing Daniele-Khrapkov method. While an exact closed form solution is obtained by factorizing a $2 \times 2$ Wiener-Hopf matrix, it is quite complicated not only algebraically, but also it contains no less than six transcendental functions. This complexity is not too surprising for this type of anisotropy. The explicit expressions of the solution can be obtained for the special case when $R_1 \cdot R_2 << 1$. Also, 4 different special cases are examined which in all, by using Fourier transform technique, the problem can be formulated into a pair of simultaneous Wiener-Hopf equations which are decoupled via a polynomial transformation and solved through the standard procedure.

II. Formulation of the Problem

A plane electromagnetic wave given by

$$\vec{E}(x, y, z) = (A_x, A_y, A_z)e^{-ik \sin \theta_y (\cos \phi_x + y \sin \phi_x)} e^{ikz \cos \theta_z}$$

(3)

and satisfying

$$\vec{k} \cdot \vec{A} = 0$$

(4)

is incident upon the $y = 0$ plane where the negative half of which ($x < 0$) is PEC, while the positive half ($x > 0$) has anisotropic conductivity at oblique incidence. The time dependence is assumed as $\exp(-i\omega t)$ and the $z$-dependence as $\exp(ik_z)$ of the incident field, which are common to all field
Maxwell’s equations, and in order to determine the representation for \( E_y \), \( \text{div} \, \vec{D} = 0 \) will be used.

For \( E_z^s \) and \( E_y^s \) which satisfy the reduced wave equation of the half-plane, one can assume the following integral representations:

\[
E_z^s = \int_L A_\pm (\alpha) e^{i \alpha z \pm i \Gamma (\alpha) y} d\alpha + E_z^s \left( E_z^s \right), \quad y \geq 0 \tag{8}
\]

\[
E_y^s = \int_L B_\pm (\alpha) e^{i \alpha z \pm i \Gamma (\alpha) y} d\alpha + E_y^s \left( E_y^s \right), \quad y \geq 0 \tag{9}
\]

where \( \Gamma (\alpha) = \sqrt{N^2 - \alpha^2} \) with \( N = \sqrt{k^2 - k_z^2} = k \sin \theta_0 \). The square root function \( \Gamma (\alpha) \) is defined in the complex \( \alpha \)-plane cut as shown in Fig.-2, such that \( \Gamma (0) = N \).

The half-plane which assumed to have a finite conductivity of \( R_1 \) in the \( x \)-direction and of \( R_2 \) in the \( z \)-direction can be characterised by the following general anisotropic resistivity conditions given by Senior[3]:

\[
\hat{y} \times \left[ \vec{E} (x, +0) - \vec{E} (x, -0) \right] = 0 \quad , \quad x > 0 \tag{5}
\]

\[
\hat{y} \times \left[ \vec{E} \times \vec{H} (x, +0) \right] = -\overrightarrow{\nabla} \times \left[ \vec{H} (x, +0) - \vec{H} (x, -0) \right] \quad , \quad x > 0 \tag{6}
\]

where \( \hat{y} \) is the unit vector directed along the \( y \)-axis. Here, \( \vec{E} \) and \( \vec{H} \) denote the total fields which are written as the sum of the incident and scattered field components

\[
\vec{E} \left( \vec{H} \right) = \vec{E}^i \left( \vec{H}^i \right) + \vec{E}^s \left( \vec{H}^s \right) \tag{7}
\]

for all \( y \).

As is known, to obtain the scattered fields, it is sufficient to consider the \( x \)- and \( z \)-components of the electric field. In order to be able to use the above boundary conditions, the tangential components of the magnetic field must also be known. These components can be derived easily from \( E_z^s \) and \( E_y^s \) via

Figure-1. Geometry of the problem.

Figure-2. Complex \( \alpha \)-plane and position of integration line \( L \), where the regularity band is determined by Im (\( \alpha \)) < Im (\( N \)) and Im (\( \alpha \)) > Im (\( N \cos \phi_0 \)).

The terms \( E_z^s \) and \( E_y^s \) are defined by

\[
E_z^s \left( E_z^s \right) = R_x \left( T_z \right) A_x^s e^{-ik_x x \pm ik_y y}, \quad y \geq 0 \tag{10}
\]

\[
E_y^s \left( E_x^s \right) = R_y \left( T_x \right) A_x^s e^{-ik_x x \pm ik_y y}, \quad y \geq 0 \tag{11}
\]

In the eqs. (8 – 11), the \( (+) \) signs together with the reflection terms are used in \( y > 0 \) half space while the \( (–) \) signs together with the transmission terms are used in \( y < 0 \) half space. \( R_x \left( T_z \right) \) and \( R_y \left( T_x \right) \) denote the reflection(transmission) coefficients related to the \( x \) and \( z \) components of the electric field that would be reflected(transmitted) if the whole plane \( y = 0 \) were characterized by a constant surface resistance \( R \).

By using Maxwell's equations and eqns. (8 – 11), \( H_x^s \) and \( H_z^s \) can now be obtained as.
In the above expressions, $\Phi_{t,2}^+ (\alpha)$ and $\Phi_{t,2}^- (\alpha)$ are yet unknown functions regular in the half-plane $\Im \alpha > \Im k_x$ and $\Im \alpha < \Im N$, respectively.

The elimination of $A_+ (\alpha)$ and $B_+ (\alpha)$ between (17 - 22) gives a matrix Wiener-Hopf equation written in the strip $\Im k_x < \Im \alpha < \Im N$ as follows:

$$\mathbf{G}(\alpha) \Phi^+(\alpha) = \Phi^-(\alpha) + \mathbf{V}(\alpha)$$

(25)

where

$$\mathbf{G}(\alpha) = \begin{bmatrix} \frac{2ak_z}{\omega \mu \Gamma} & -\frac{2N^2}{\omega \mu \Gamma} + \frac{1}{R_1} \\ -\frac{2(k^2 - \alpha^2)}{\omega \mu \Gamma} & \frac{1}{R_2} & \frac{2ak_z}{\omega \mu \Gamma} \end{bmatrix}$$

(26)

and

$$\mathbf{V}(\alpha) = \begin{bmatrix} -h \\ -j \end{bmatrix}$$

(27)

where $h$ and $j$ are given by eqs. (23) and (24). Here, $\mu$ denotes the magnetic permeability of the surrounding medium. $\Phi^+(\alpha)$, $\Phi^-(\alpha)$ and $\mathbf{V}(\alpha)$ are column vectors, where the terms $\Phi^+(\alpha)$, $\Phi^-(\alpha)$ are unknown functions that would be determined later and $\mathbf{V}(\alpha)$ corresponds to the contributions of the incident and reflected fields as given in (27).

III. Solution of the Wiener-Hopf System

The formal solution is derived for the diffraction problem by employing Daniele-Khrapkov method. While an exact closed form solution is obtained by factorizing a $2 \times 2$ Wiener-Hopf matrix, it is quite complicated not only algebraically, but also it contains no less than six transcendental functions. This complexity is not too surprising for this type of anisotropy[4]. The explicit expressions of the solution can be obtained for the special case when $R_1 \cdot R_2 << 1$. Also, 4 different special cases (Case for $R_2 \to \infty$, Case for $R_1 \to \infty$, Case for $R_1 = 0$, and Case for $R_2 = 0$) are examined which in all, by using Fourier transform technique, the problem can be formulated into a pair of simultaneous Wiener-Hopf equations which are decoupled via a polynomial transformation and solved through the standard procedure. Here, in this paper we are going to give the solution of only one special case of the scalar Wiener-Hopf system; i.e. case for $R_2 \to \infty$.

Letting $R_2 \to \infty$, the problem reduces to two simultaneous Wiener-Hopf equations:

$$\frac{2ak_z}{\omega \mu \Gamma} \Phi^+_t (\alpha) + \left[ -\frac{2N^2 + \omega \mu \Gamma}{R_1} \right] \Phi^+_t (\alpha) = \Phi^+_t + \mathbf{V}_t$$

(28)

and

$$\frac{1}{\omega \mu \Gamma} \left[ -2(k^2 - \alpha^2) \Phi_t^+ + 2ak_z \Phi_t^+ \right] = \Phi_t^+ + \mathbf{V}_t.$$
Since $\Phi^+_1$ and $\Phi^+_2$ are both regular functions in the upper half-plane, the sum of these two functions in (29) is also regular in the upper half-plane. Let

$$\Psi^+(\alpha) = -2(k^2 - \alpha^2)\Phi^+_1 + 2ak_s\Phi^+_2$$

(30)

So (29) is reduced into the below form;

$$\frac{1}{\omega\mu \Gamma(\alpha)} \Psi^+(\alpha) = \Phi^- + V_2$$

(31)

where $V_2$ is given by (27).

As is seen, the polynomial transformation given in (30) reduced the simultaneous system of equations in (28 - 29) into two scalar Wiener-Hopf equations. Therefore, first eq. (31) will be solved and $\Phi^+_1$ is expressed in terms of $\Phi^+_2$. Then, eq. (28) will also be reduced to a scalar equation involving only $\Phi^+_2$ and $\Phi^+_1$.

Eqn. (31) is a scalar Wiener-Hopf equation whose solution is

$$\Phi^+_1(\alpha) = \frac{ak_s}{(k^2 - \alpha^2)}\Phi^+_2(\alpha) - \frac{1}{2(k^2 - \alpha^2)} \Psi^+(\alpha)$$

(32)

with

$$\Psi^+(\alpha) = \frac{-jkz_0}{2\pi i(\alpha - k_s)}\sqrt{N - k_s\sqrt{N + \alpha}}$$

(33)

Now, substituting this in eqn. (28) and rearranging yields

$$\frac{T(\alpha)}{\omega\mu \Gamma(\alpha)(k^2 - \alpha^2)} \Phi^+_2 = \Phi^+_1 + M(\alpha)$$

(34)

where

$$T(\alpha) = \frac{kz_0 \sigma_1 \sigma_2}{R_1 N^2} \cdot \frac{\Gamma^3}{\chi(\xi_1, \alpha) \chi(\xi_2, \alpha)}$$

(35)

with

$$\xi_1 = N/\sigma_1 \quad \text{and} \quad \xi_2 = N/\sigma_2$$

(36)

and

$$\sigma_{1,2} = -\frac{kR_1}{Z_0} \pm k\sqrt{\left(\frac{R_1}{Z_0}\right)^2 - \cos^2 \theta_0}$$

(37)

and where

$$M(\alpha) = V_1 + \frac{ak_s}{\omega \mu \Gamma(\alpha)(k^2 - \alpha^2)} \Psi^+(\alpha)$$

(38)

with $V_1$ given by (27). The function $\chi(\xi, \alpha)$ given by eqn. (35) is defined as

$$\chi(\xi, \alpha) = \frac{\Gamma(\alpha)}{N + \xi \Gamma(\alpha)} = \chi^+(\xi, \alpha) \chi^-(\xi, \alpha)$$

(39)

which is factorized in terms of Maliuzhinetz function[5]. Then performing Wiener-Hopf decomposition, one obtains

$$\Phi^+_2 = \chi^-(\xi_1, k_s) \chi^-(\xi_2, k_s) \chi^+(\xi_1, \alpha) \chi^+(\xi_2, \alpha)$$

$$\frac{(k - k_s)^2 \sin^2 \theta_0 R_1 (k + \alpha)}{\cos^2 \theta_0 (N + \alpha)} h \frac{\sin^2 \theta_0 \sqrt{R_1 (k + \alpha)}}{2\pi i(\alpha - k_s)(k + k_s)}$$

(40)

Since $\Phi^+_1$ were expressed in terms of $\Psi^+$ and $\Phi^+_2$ in (32), by using (33)

$$\Phi^+_1 = \frac{jkz_0\sqrt{N - k_s \sqrt{N + \alpha}}}{4\pi i(\alpha - k_s)(k^2 - \alpha^2)} + \frac{ak_s}{(k^2 - \alpha^2)} \Phi^+_2$$

(41)

is obtained. Since $\Phi^+_1(\alpha)$ and $\Phi^+_2(\alpha)$ are now completely determined, the spectral coefficients $A_\pm$ and $B_\pm$ can be written from (17-20) to give

$$A_\pm(\alpha) = \Phi^+_1(\alpha),$$

(42)

$$B_\pm(\alpha) = \Phi^+_2(\alpha).$$

(43)

IV. Conclusions

Now, using the expressions of the spectral coefficients, the diffracted fields given by (8) and (9) can be obtained by using Steepest Descent method:

$$E^*_2(\rho, \phi) \sim D_\pm(\theta_0, \phi_0, \phi) \frac{\sin N\phi}{\sqrt{N\rho}}$$

(44)

where

$$D_\pm(\theta_0, \phi_0, \phi) = \frac{\frac{e^{-i\phi/4}}{\sqrt{2\pi}} \chi(\xi_1, N \cos \phi) \chi(\xi_2, N \cos \phi)}{(1 - \cos \phi) \cos \theta_0 \cos \phi_0}$$

$$\chi(\xi_1, N \cos \phi) \chi(\xi_2, N \cos \phi_0) (1 - \sin \theta_0 \cos \phi_0) \sin \phi$$

$$\cos \phi + \cos \phi_0 \cos \phi_0$$

(45)

with

$$Y = \left\{ -\sin \theta_0 - \sin \theta_0 \sin^2 \phi_0 \right\}$$

$$\cos \theta_0 \sin \phi_0$$

$$+ \sin \theta_0 \sin \phi_0 \cos \phi_0 + \cos \theta_0 \cos \phi_0 \right\}$$

$$\left\{ -2 \sin^2 \theta_0 \sin^2 \phi_0 \right\}$$

$$\cos \theta_0 + \sin \theta_0 \cos \theta_0 \cos \phi_0 - \sin \theta_0 \cos \theta_0 \sin \phi_0 \right\}$$

$$\frac{\cos \phi_0}{(1 + \sin \theta_0 \cos \phi_0) \sin \theta_0 \sin \phi_0}$$

(46)

Here, $\chi(\xi, \alpha)$ is the function given in (39) and $\xi_{1,2}$ are defined as follows:

$$\xi_{1,2} = \frac{\sin \theta_0}{\frac{R_1}{Z_0} \pm \sqrt{\left(\frac{R_1}{Z_0}\right)^2 - \cos^2 \theta_0}}$$

(47)

Also, in a similar manner

$$E^*_2(\rho, \phi) \sim D_\pm(\theta_0, \phi_0, \phi) \frac{\sin N\phi}{\sqrt{N\rho}}$$

(48)

is obtained, where

$$D_\pm(\theta_0, \phi_0, \phi) = \frac{\frac{e^{-i\phi/4}}{\sqrt{2\pi \sin \phi_0} \cos \phi + \cos \phi_0}}{(1 - \sin^2 \theta_0 \sin^2 \phi)}$$

$$\frac{\sqrt{1 - \cos \phi}}{\sqrt{1 - \cos \phi}} \cdot \left\{ -2 \sin^2 \theta_0 \sin^2 \phi_0 \right\}$$

(49)
\[
\frac{-\cos^2 \theta_0 + \sin \theta_0 \cos \theta_0 \cos \phi_0 - \sin \theta_0 \cos \theta_0 \sin \phi_0}{\sin \theta_0 \cos \phi_0 \cos \theta_0 \frac{D_2}{1 - \sin^2 \theta_0 \cos^2 \phi}} \cdot \frac{D_2}{D_2} (\theta_0, \phi_0, \phi).
\]

with \( D_2 (\theta_0, \phi_0, \phi) \) given by (45-47).

Some numerical results have been obtained for the diffracted fields where the ambient medium was taken as free space. The incident plane wave field strength was assumed to be 1 V/m. The diffracted field expressions involve the split functions \( \chi^\pm (\alpha) \), which can be written in terms of the Maluzhinetz functions, as is done by Uzgoren et al.[5]. By using approximate formula given by Volakis and Senior[6], the Maluzhinetz functions and the diffracted fields are computed.

![Figure-3. Variation of the diffraction coefficient \((20 \log_{10} D_2^2)\) with respect to the observation angle for different incidence angles and different resistivities.](image1.png)

![Figure-4. Variation of the diffraction coefficient \((20 \log_{10} D_2^2)\) with respect to the observation angle for different incidence angles and different resistivities.](image2.png)

Figures 3-4 illustrate the variation of the amplitude of the -component of the diffracted field \(20 \log_{10} \left| u_4 \right| \times \sqrt{N \rho} \) by the observation angle for the different values of the normalized resistance \( R/Z_0 \). The diffracted field expressions given are not uniform and it is expected that the field will vary very large values in the transition regions. As seen from the Figs., the transition boundaries are determined by the incidence angle as \((\pi - \phi_0)\) and \((\pi + \phi_0)\).

The problem of diffraction from a discontinuity formed by a PEC and an anisotropic resistive half-planes is considered for this first time in this work. Therefore there has been no opportunity to compare the results on Figs. 3-4 with some previously obtained results to validate the accuracy of our high-frequency solution. But it should be noted that, for \( R_1 \) (\( R_2 \)) = 0, the -component (z-component) of the scattered field involves only the reflected term as expected and this may validate the accuracy of the solution.

V. References