Homework 7

Problem 14.5 in the textbook. In addition to parts a-d, at \( r = 1 \) cm compute the e) shear stress \( \tau \) and f) shear strain \( \gamma \).

Answers e) \( \tau = 119.37 \) MPa f) \( \gamma = 0.683^\circ \)

Problem 2
A circular tube subjected to pure torsion by moments \( M \) has an outer radius \( r_o \) equal to twice the inner radius \( r_i \).

A. If the maximum shear strain is measured as \( 400 \times 10^{-6} \) rad, what is the shear strain \( \gamma_{\text{min}} \) at the inner surface?

B. If the maximum allowable rate of twist is \( 0.45^\circ /m \), and the maximum shear strain is to be kept at \( 400 \times 10^{-6} \) rad by adjusting the moment \( M \), what is the minimum required outer radius \( r_o \)?

Answers A. \( \gamma_{\text{min}} = 200 \times 10^{-6} \) rad B. minimum \( r_o = 50.93 \) mm

Problem 3
A circular tube of length \( L = 0.90 \) m is loaded in torsion by moments \( M \).

A. If the inner radius is \( r_i = 40 \) mm and the measured angle of twist between the ends is \( 0.5^\circ \), what is the shear strain \( \gamma_{\text{min}} \) at the inner surface?

B. If the maximum allowable shear strain is \( 0.0005 \) rad and the angle of twist is kept at \( 0.5^\circ \) by adjusting the moment \( M \), what is the maximum permissible outer radius \( r_o \)?

Answers A. \( \gamma_{\text{min}} = 387.9 \times 10^{-6} \) rad B. maximum \( r_o = 51.6 \) mm
Problem 4
A circular tube is loaded in torsion by moments \( M \). The tube is 30 cm long with inside and outside diameters of 6 and 8 cm, respectively. When the moment is 5800 N-m, the measured twist angle is 3.63°.

Compute the maximum shear strain \( \gamma_{\text{max}} \), the shear modulus of elasticity \( G \), and the maximum shear stress \( \tau_{\text{max}} \).

\[ \begin{align*}
\gamma_{\text{max}} & = 8.45 \times 10^{-3} \text{ rad} \\
G & = 10.0 \text{ GPa} \\
\tau_{\text{max}} & = 84.4 \text{ MPa}
\end{align*} \]

Problem 5
A solid steel bar of circular cross section has a diameter \( d = 1.5 \) inches, length \( L = 4.5 \) feet, and shear modulus of elasticity \( G = 11.5 \times 10^6 \) psi. The bar is subjected to moments \( M \) at the ends.

A. If the moments have magnitude \( M = 250 \) lb-ft, what is the maximum shear stress in the bar? What is the angle of twist at the ends?
B. If the allowable shear stress is 6000 psi and the allowable angle of twist is 2.5°, what is the maximum allowable moment \( M \)?

\[ \begin{align*}
\text{Answers A. } & \quad \tau_{\text{max}} = 4527.1 \text{ psi} \quad \theta = 1.62° \\
& \quad \text{B. } M_{\text{max}} = 331.4 \text{ lb-ft}
\end{align*} \]

Problem 6
Determine the minimum required diameter of a solid circular bar given the design requirements:

i. maximum allowable shear stress is 70 MPa;
ii. twist angle between two points located 2.5 m apart must not exceed 3°;
iii. applied moment is 1400 N-m;
iv. shear modulus \( G = 11 \) GPa.

\[ \text{Answers minimum diameter is 88.7 mm} \]
Problem 7
Our textbook defines torsional stiffness $k_T$ as the ratio of the applied torque to the resultant angular deformation (the angle of twist). Using Equation 14.29 we can show:

$$\frac{M}{\theta} = k_T = \frac{GJ}{L}$$

where $G$ is the shear modulus of elasticity, $J$ is the polar moment of inertia, and $L$ is the length of the rod.

A. For the rod shown with dimensions $L = 1.2$ m, $r_o = 15$ mm and shear modulus of elasticity $G = 28$ GPa:
   i. compute the torsional stiffness;
   ii. compute the maximum shear strain, maximum shear stress, and the applied moment (torque) if the angle of twist is 4 degrees;
   iii. compute the shear strain and shear stress at $r = 6$ mm.
   iv. Sketch the shear stress and shear strain versus the distance from the center of the rod.

B. The center of the rod is now drilled out to create a hollow tube with inner diameter $r_i = 12$ mm.
   i. Derive the polar moment of inertia for the hollow tube:

   $$J = \frac{\pi}{2} (r_o^4 - r_i^4)$$

   (This is the equation above Equation 14.27 in our book.)
   ii. Compute the torsional stiffness for the hollow tube.
   iii. Compute the maximum and minimum shear strain, the maximum and minimum shear stress, and the twist angle if the applied moment is same as in part A.ii above.
   iv. Compute the shear strain and shear stress at $r = 14$ mm.
   v. Sketch the shear stress and shear strain versus the distance from the center of the hollow tube.

Answers
A. $k_T = 1855.5$ N-m; $\gamma_{\text{max}} = 8.73 \times 10^{-4}$, $\tau_{\text{max}} = 24.4$ MPa, $M = 129.5$ N-m;
   at $r = 6$ mm: $\gamma = 3.49 \times 10^{-4}$, $\tau = 9.77$ MPa

B. $k_T = 1095.5$ N-m; $\gamma_{\text{max}} = 14.8 \times 10^{-4}$, $\gamma_{\text{min}} = 11.8 \times 10^{-4}$,
   $\tau_{\text{max}} = 41.4$ MPa, $\tau_{\text{min}} = 33.1$ MPa, $\theta = 6.78^\circ$;
   at $r = 14$ mm: $\gamma = 1.38 \times 10^{-3}$, $\tau = 38.6$ MPa
Problem 8
A solid shaft and a hollow tube are constructed of the same material (with shear modulus $G$ and density $\rho$) and have the same length $L$ and outer radius $r_o$. The hollow tube inner radius is $0.6r_o$.

A. Assuming the shaft and tube are subjected to the same moment, compare their maximum shear stresses $\tau_{\text{max}}$, angles of twist $\theta$, and weights $W$.
B. Calculate the ratio (tube/shaft) for $\tau_{\text{max}}$, $\theta$, and $W$.

Answers
A. $\tau_{\text{shaft}} = \frac{2M}{\pi r_o^3}$, $\theta_{\text{shaft}} = \frac{2ML}{\pi Gr_o^4}$, $W_{\text{shaft}} = \rho L\pi r_o^2$
B. $\tau_{\text{tube}}/\tau_{\text{shaft}} = 1.15$, $\theta_{\text{tube}}/\theta_{\text{shaft}} = 1.15$, $W_{\text{tube}}/W_{\text{shaft}} = 0.64$

Problem 9
As the premier biomedical engineer at your company, you have been chosen to design the artificial tibia for a new prosthetic leg. This artificial bone will be a circular shaft that must support a torque of 1200 N-m without exceeding a maximum shear stress of 40 MPa. The shear modulus of the shaft material is 78 GPa.

The shaft can be either a solid bar or a hollow tube; both have a circular cross section.

A. Compute the required diameter $d_o$ of the solid bar.
B. Compute the required outer diameter of the hollow tube if the thickness of the tube is one-eighth of the outer radius.
C. Determine the outer diameter ratio and weight ratio of the hollow tube to the solid bar.
D. Compute the maximum and minimum shear stress and shear strain in each shaft.
E. Compute the torsional stiffness ratio of the tube to the bar.
F. Compute the rate of twist in each shaft when a moment of 160 N-m is applied.
G. Given these results, which is “stronger,” the solid bar or hollow tube? If maximal strength and minimal weight are design criteria, should the prosthetic be a solid bar or tube? What does this imply for the geometric structure of long bones?

Answers
A. solid bar: $d_o = 5.35$ cm
B. hollow tube: $d_o = 7.17$ cm
C. tube/bar diameter ratio = 1.342, weight ratio = 0.422
D. bar: $\tau_{\text{max}} = 40$ MPa, $\tau_{\text{min}} = 0$, $\gamma_{\text{max}} = 0.513 \times 10^{-3}$, $\gamma_{\text{min}} = 0$
   tube: $\tau_{\text{max}} = 40$ MPa, $\tau_{\text{min}} = 35$ MPa, $\gamma_{\text{max}} = 0.513 \times 10^{-3}$, $\gamma_{\text{min}} = 0.449 \times 10^{-3}$
E. tube/bar $k_T$ ratio = 1.342
F. tube: $0.109^\circ$/m, bar: $0.147^\circ$/m
Problem 10
The picture shows midshaft cross sections of femurs from a woman in her thirties (left) and from a woman in her nineties (right) [RB Martin, DB Burr, NA Sharkey; Skeletal Tissue Mechanics (Springer 1998)]. Osteoporosis has greatly reduced the cross sectional area of the older specimen.

For this analysis the femur will be modeled as a hollow cylindrical tube with an outer diameter of 6 cm and ultimate stress of 130 MPa. A moment is applied to the femur so the load is pure torsion.

A. Assume the healthy femur has an inner diameter of 3 cm. Compute the applied moment required to break the healthy femur.

B. Due to osteoporosis the cortical thickness of the older femur is reduced by 50%. Compute the moment required to break the older femur.

C. Sketch the shear stress versus the distance from the center of the long axis for both femurs.

D. Sketch the shear strain versus the radius for both femurs. Assume $G = 3.3$ GPa.

E. In light of the above results, why do older people suffer more bone fractures?

Answers
A. $M = 5169$ N-m
B. $M = 3769$ N-m
C. younger: $\tau_{\min} = 65$ MPa, older: $\tau_{\min} = 97.5$ MPa
   both: $\tau_{\max} = 130$ MPa
Problem 11
While out jogging, a clumsy professor misjudges a curb and begins to fall. Due to his stride, the body weight generates a compressive axial force of 1725 N on the femur. During the fall, a twisting torque of $-900$ N-m is applied to the femur. The bone has inner and outer diameters of 4.4 cm and 5.0 cm, respectively. For the following analysis consider a typical material element on the anterior surface of the femur.

A. Compute the axial stress $\sigma_Y$ caused by the body weight.
B. Compute the torsional shear stress $\tau_S$ caused by the twisting torque of the fall. (When discussing principal stresses we’ve been calling this $\tau_{\text{max}}$, but part D will show this isn’t the maximum shear stress.)
C. Compute the principal stresses that arise from the combined load of the axial force and the twisting torque. Also compute the orientation of the principal plane.
D. Compute the maximum shear stress, the associated normal stresses, and orientation of the maximum shear plane.
E. Experimental data from the human femur give the ultimate stress values below. Since the combined load is rapidly applied, the ultimate stress approximates the failure stress. Given these data, how does the bone fracture? Sketch the crack path on the shaft of the femur.

<table>
<thead>
<tr>
<th>load type</th>
<th>$\sigma_U$ (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>tension</td>
<td>78.8–144</td>
</tr>
<tr>
<td>compression</td>
<td>170–209</td>
</tr>
</tbody>
</table>

Answers
A. $\sigma_Y = -3.89$ MPa
B. $\tau_S = -91.60$ MPa
C. $\sigma_1 = 89.7$ MPa, $\sigma_2 = -93.6$ MPa, $\theta_P = -44.4^\circ$
D. $\tau_{\text{max}} = 91.62$ MPa, $\sigma_{\text{avg}} = -1.95$ MPa, $\theta_S = -89.4^\circ$

This combined loading problem demonstrates the superposition of the individual loads. In most analyses superposition is justified as long as we consider only linear elasticity. Since the stresses here are rather high, and the loads are applied rapidly, we are actually in the realm of nonlinear elasticity so superposition really doesn’t hold. But it’s a fun problem to think about even if the assumptions are a bit of a “stretch” . . .
Problem 12
Suppose you are designing an artificial femur, and you need to increase its torsional stiffness compared to an older model. From Problem 7 and the textbook’s Equation 14.29 we know the torsional stiffness is

\[ k_T = \frac{M}{\theta} = \frac{GJ}{L} \]

where \( M \) is the applied moment, \( \theta \) is the twist angle, \( G \) is the shear modulus of elasticity, \( J \) is the polar moment of inertia, and \( L \) is the length of the femur.

Your new prosthetic femur will replace a solid circular cylinder with a polar moment of inertia \( J_{old} \); your new femur must have \( J_{new} = 2J_{old} \). The material will not change. Since you know hollow tubes are “more stiff” than solid cylinders, you decide to use a circular tube with inner radius \( r_i \) and outer radius \( r_o \). The cross-sectional area of your new femur must be same as that of the old femur, which has radius \( r \).

A. Determine \( r_i \) and \( r_o \) in terms of \( r \).
B. What is the weight ratio of the two femurs?

Given the data

\[ M = 7.6 \text{ N-m} \quad r = 6 \text{ mm} \quad L = 45.7 \text{ cm} \quad G = 28 \text{ GPA} \]

for both femurs compute:
C. twist angles and the maximum and minimum shear stresses
D. torsional stiffness \( k_T \)

**Answers**

A. \( r_i = r / \sqrt{2} \approx 0.707r \quad r_o = \sqrt{3/2}r \approx 1.255r \)

C. old: \( \theta = 3.5^\circ \quad \tau_{max} = 22.4 \text{ MPa} \quad \tau_{min} = 0 \text{ MPa} \)
new: \( \theta = 1.75^\circ \quad \tau_{max} = 13.7 \text{ MPa} \quad \tau_{min} = 7.92 \text{ MPa} \)

D. old: \( k_T = 125 \text{ N-m} \quad \text{new: } k_T = 250 \text{ N-m} \)