2^{nd} Order Transients – 3

the full solution

How to find the constants?

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + x_\infty$$

$$x(t) = B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t + x_\infty$$

– x_∞ is easy
  - Short inductor, open capacitor, solve DC problem

– Initial condition ($v_C(0)$ or $i_L(0)$) is not enough

$$x(0) = A_1 + A_2 + x_\infty$$

$$x(0) = B_1 + x_\infty$$

Approach: include a derivative condition at time 0

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + x_\infty$$

$$x'(t) = s_1 A_1 e^{s_1 t} + s_2 A_2 e^{s_2 t}$$

$$x'(0) = s_1 A_1 + s_2 A_2$$

use with $x(0) = A_1 + A_2 + x_\infty$

$$x(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t + x_\infty$$

$$x'(t) = -\alpha B_1 e^{-\alpha t} \cos \omega_d t - \alpha B_2 e^{-\alpha t} \sin \omega_d t$$
$$-\omega_d B_1 e^{-\alpha t} \sin \omega_d t + \omega_d B_2 e^{-\alpha t} \cos \omega_d t$$

$$x'(0) = -\alpha B_1 + \omega_d B_2$$

use with $x(0) = B_1 + x_\infty$

Solving

- Overdamped (2 real roots, s_1, s_2):

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + x_\infty$$

$$A_1 = \frac{x'(0) - s_2[x(0) - x_\infty]}{s_1 - s_2}$$

$$A_2 = \frac{s_1[x(0) - x_\infty] - x'(0)}{s_1 - s_2}$$

- Underdamped (complex roots, $-\alpha \pm j\omega_d$):

$$x(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t + x_\infty$$

$$B_1 = x(0) - x_\infty$$

$$B_2 = \frac{x'(0) + \alpha[x(0) - x_\infty]}{\omega_d}$$

Question – how do we get the derivative's value?

- Answer – Recall

For a capacitor

$$i = C\,\frac{dv}{dr}$$

$$\text{so } v'(0) = \frac{1}{C}\,i(0)$$

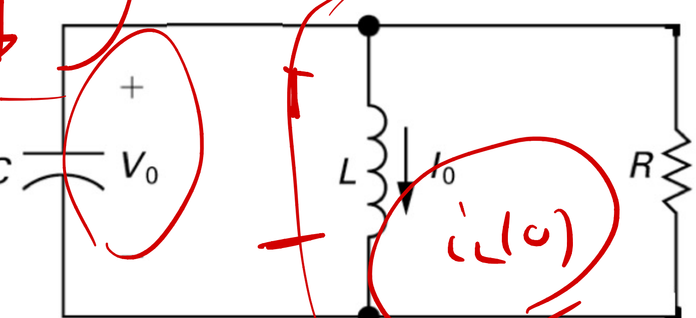
For an inductor

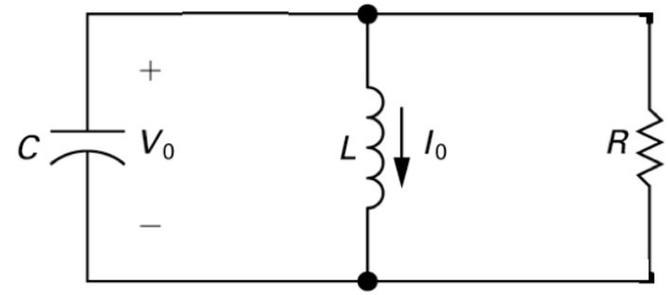
$$v = L\,\frac{di}{dr}$$

$$\text{so } i'(0) = \frac{1}{L}\,v(0)$$

Use KVL, KCL to get the capacitor current or inductor voltage

- Example – the parallel case





- Use KCL to find $v_C'(0)$

$$i_C(t) + i_R(t) + i_L(t) = 0 \quad \rightarrow \quad C\frac{dv_C(t)}{dt} + \frac{v_C(t)}{R} + i_L(t) = 0$$

$$v_C'(0) = -\frac{v_C(0)}{RC} - \frac{i_L(0)}{C}$$

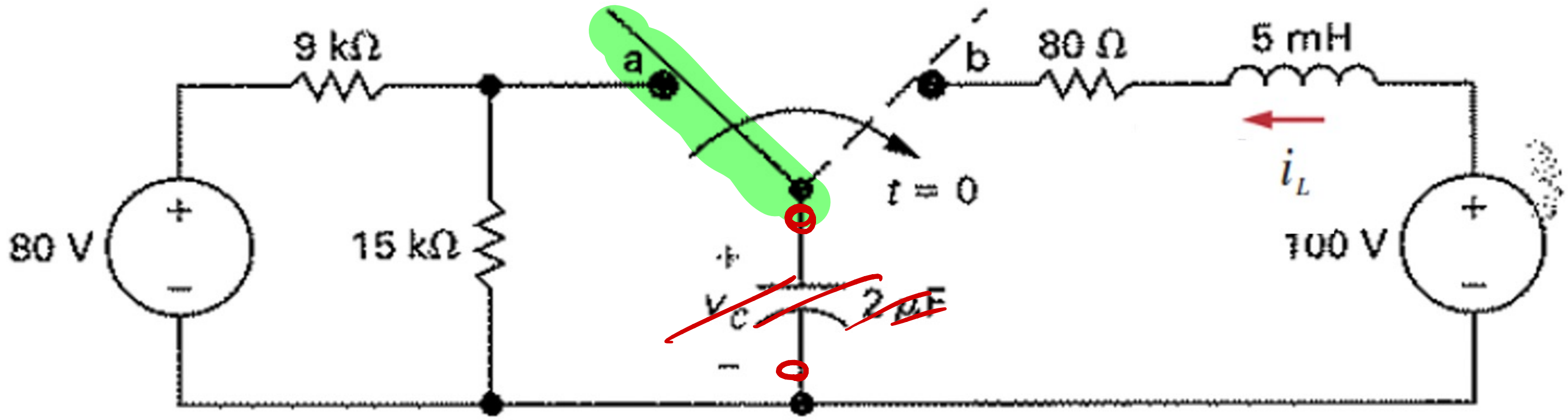
- Use KVL to find $i_L'(0)$

$$v_L(t) = v_C(t) \quad \rightarrow \quad L\frac{di_L(t)}{dt} = v_C(t)$$

$$i_L'(0) = \frac{v_C(0)}{L}$$

Example: Find the derivative conditions

$i_L'(0) = 10{,}000\ A/s$
$v_C'(0) = 0\ V/s$



$t \to \infty$

$i_L(\infty) = 0$

$v_c(\infty) = 100\ v$

$t = 0^-$

$v_c(0^-) = 50\ v$

$i_L(0^-) = 0$

$t = 0$

$v_c(0) = 50\ v$

$i_L(0) = 0$

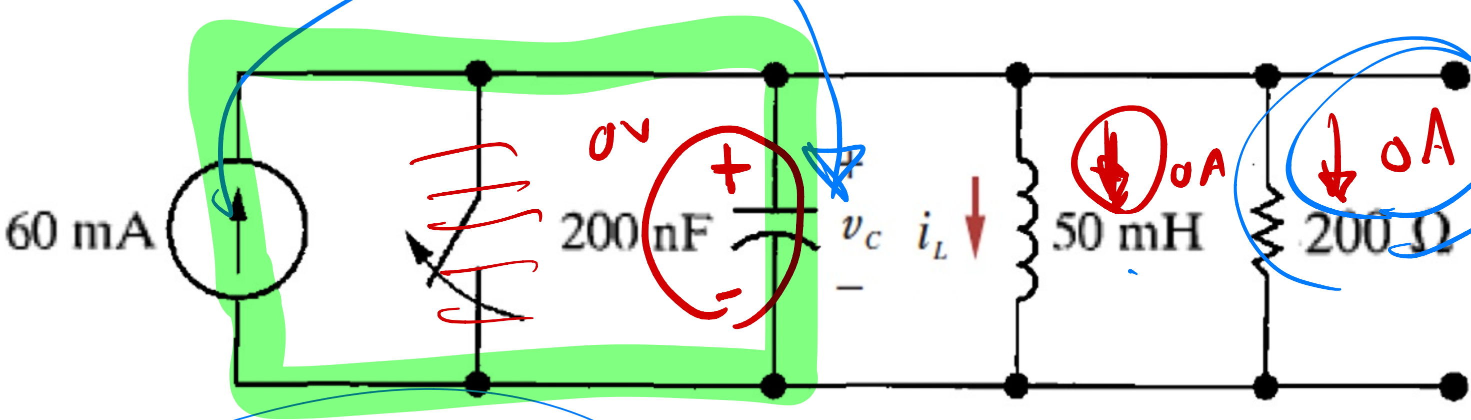
$v_c'(0) = 0$

$i_L'(0) = 10^4$

Example: Find the derivative conditions

$i_L'(0) = 0\ A/s$

$v_C'(0) = -300{,}000\ V/s$



$t = 0^-$

$v_c(0^-) = v_c(0) = 0$

$i_L(0^-) = i_L(0) = 0$

$t = 0$

$v_c'(0) = \frac{1}{C} i_c(0)$

$= \frac{1}{C}\left[60 \cdot 10^{-3}\right]$

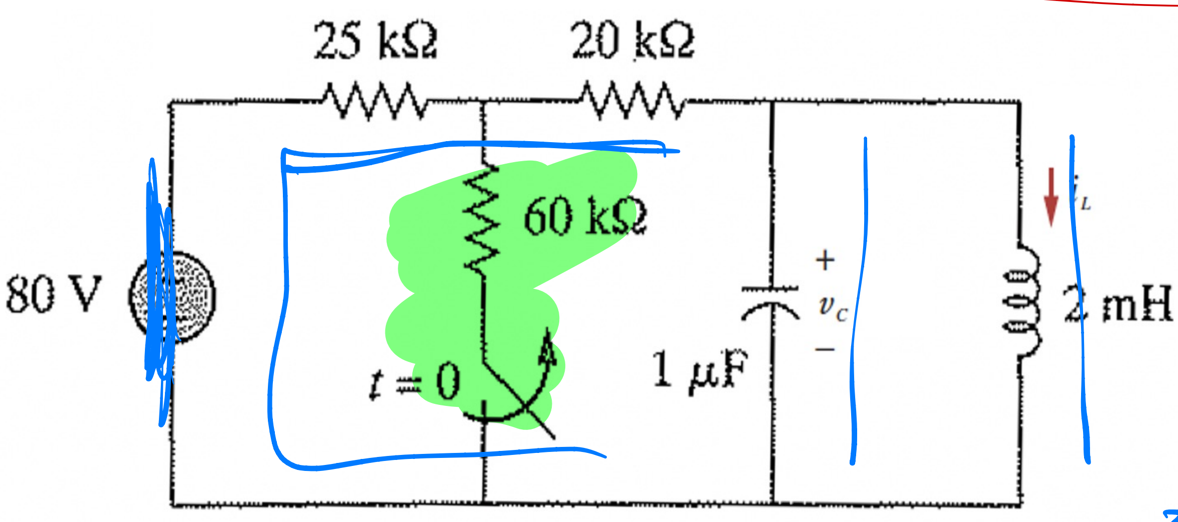
$i_L'(0) = \frac{1}{L} v_L(0) = 0$

Total Solution

1. Identify type (series/parallel) and values of R,L,C
2. Root characteristic equation, to find form
3. Find $i_L(0)$ and $v_C(0)$
4. For variable of interest, find $x(0), x'(0)$, and $x(\infty)$
5. Assemble answer

Example: Find $v_C(t)$

$$v_C(t) = -7.45\, e^{-11.1t} \sin 22361t \; V$$



parallel

$R = 45\ k\Omega$

$$s^2 + \frac{1}{RC}s + \frac{1}{LC}$$

$$s^2 + 222s + 5\cdot 10^8 = 0$$

$$s = -11.1 \pm j\,22361$$

$$v_C(t) = B_1 e^{-11.1t}\cos 22361t + B_2 e^{-11.1t}\sin 22361t + v_C(\infty)$$

($B_1 = 0$, $v_C(\infty) = 0$)

$v_C(\infty) = 0$

$V_C(0) = 0$

$v_C'(0) = \frac{5}{18}\cdot 10^6$

Cheat Sheet

Parallel:

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

Series:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

Overdamped:

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + x_\infty$$

$$A_1 = \frac{x'(0) - s_2[x(0) - x_\infty]}{s_1 - s_2} \quad A_2 = \frac{s_1[x(0) - x_\infty] - x'(0)}{s_1 - s_2}$$

Underdamped:

$$x(t) = B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t + x_\infty$$

$$B_1 = x(0) - x_\infty \quad B_2 = \frac{x'(0) + \alpha[x(0) - x_\infty]}{\omega_d}$$

Practice problem: Find the derivative conditions

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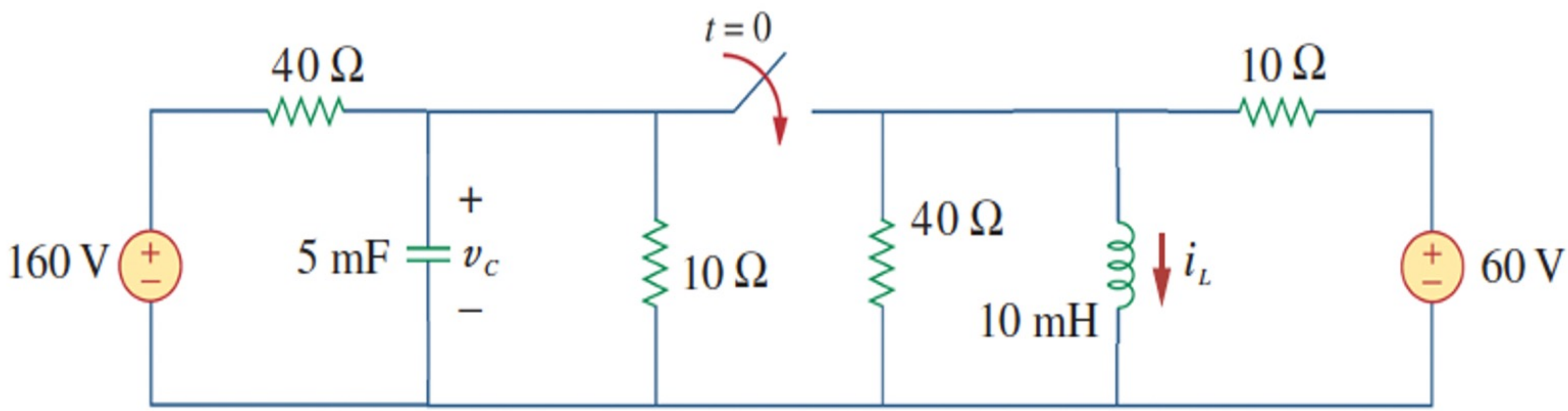
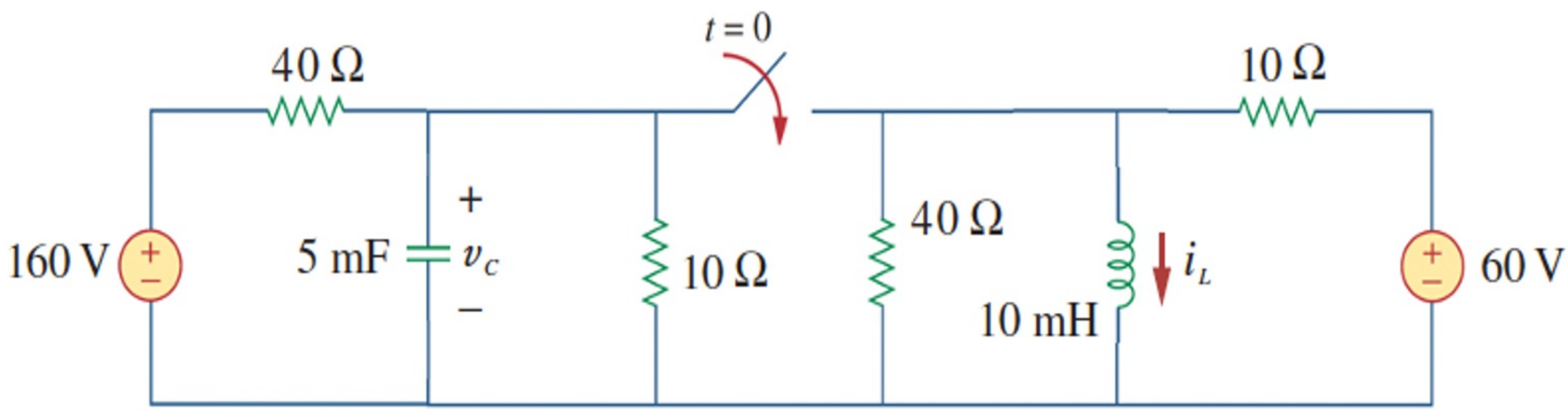


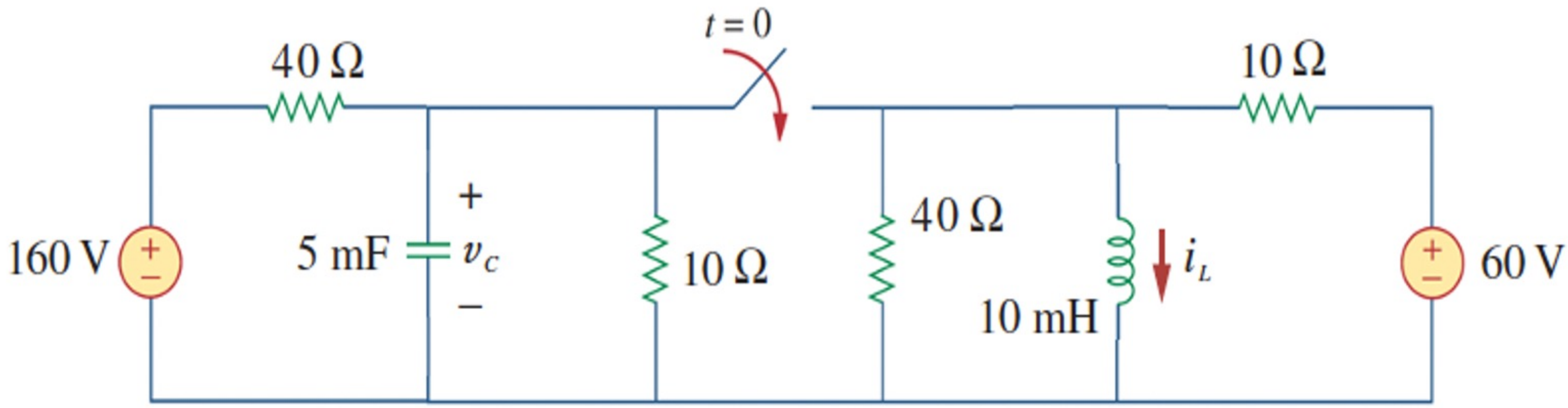
$$i_L'(0) = 3200\ A/s$$
$$v_C'(0) = -800\ V/s$$

Practice problem: Find the final form for $i_L(t)$



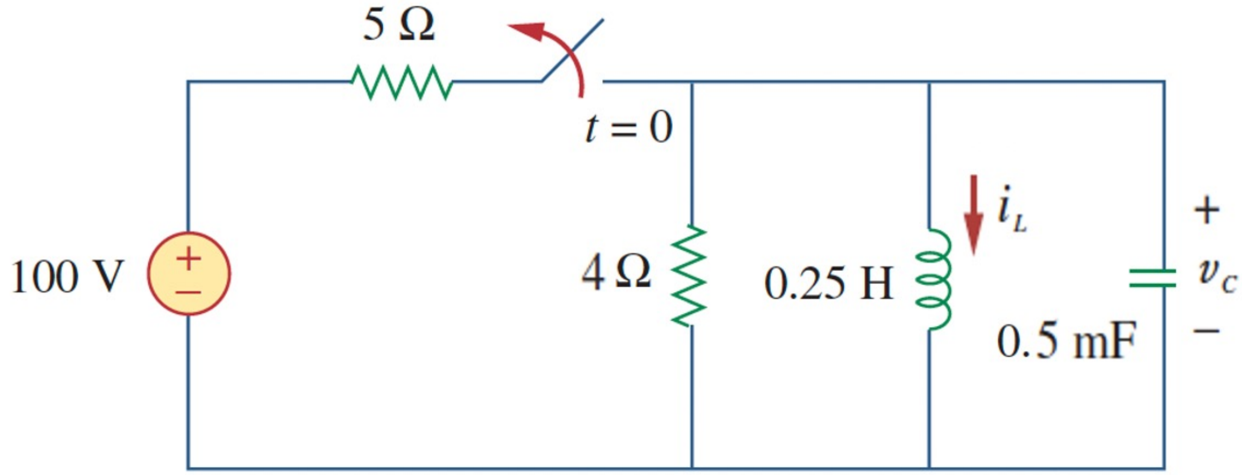
$$i_L(t) = -4\, e^{-25t} \cos 139t + 22.3.\, e^{-25t} \sin 139t + 10\ A$$

Practice problem: Find the final form for $v_C(t)$



$v_C(t) = 32\, e^{-25t} \cos 139t$ V

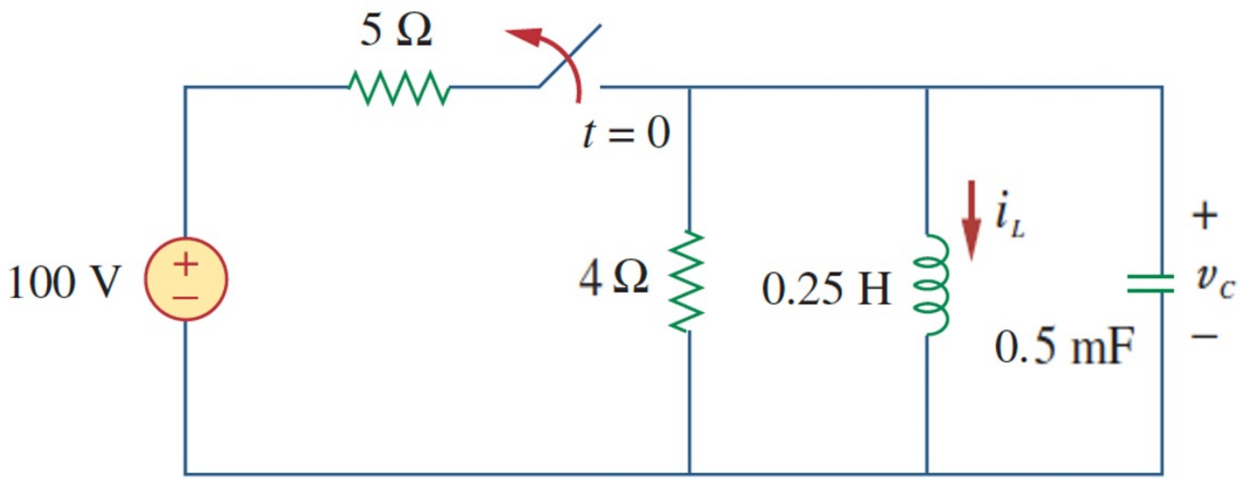
Practice problem: Find form and the initial, final, and derivative conditions



$$x(t) = A_1 e^{-16.6t} + A_2 e^{-483t} + x_\infty$$

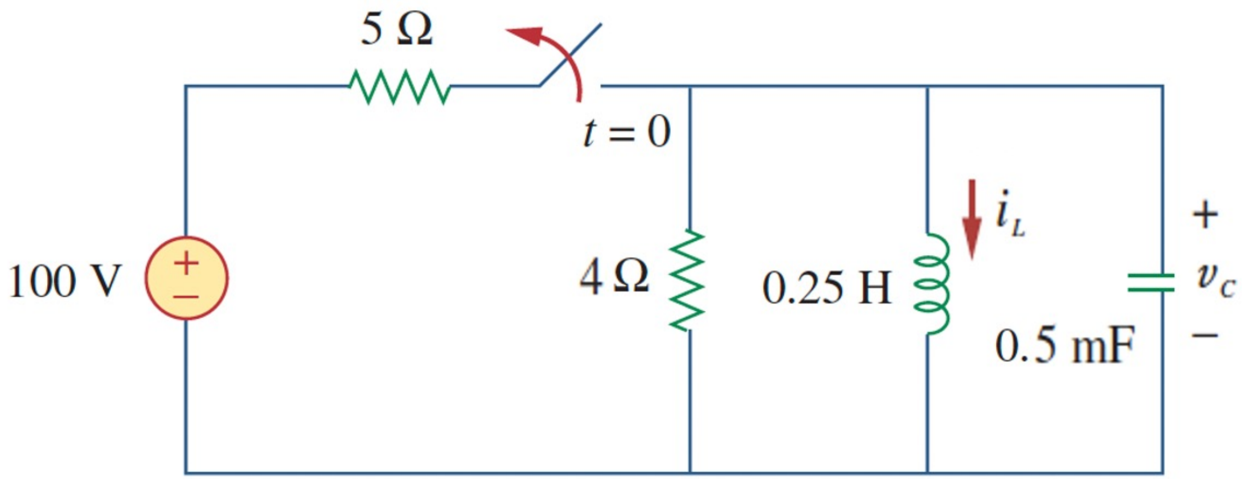
$$i_L(0) = 20\ A$$
$$i_L(\infty) = 0\ A$$
$$i_L{}'(0) = 0\ A/s$$
$$v_C(0) = 0\ V$$
$$v_C(\infty) = 0\ \mathrm{V}$$
$$v_C{}'(0) = -40{,}000\ V/s$$

Practice problem: Find the final form for $i_L(t)$



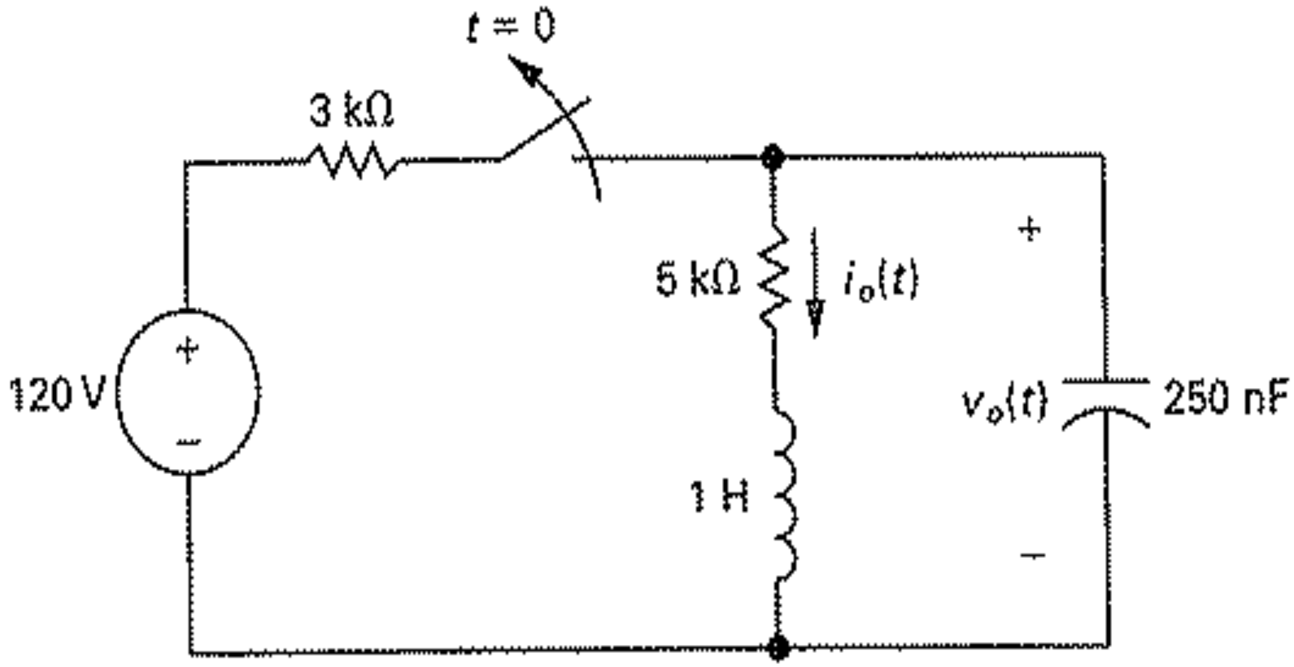
$$i_L(t) = 20.7e^{-16.6t} - 0.709\, e^{-483t}\,\text{A}$$

Practice problem: Find the final form for $v_C(t)$



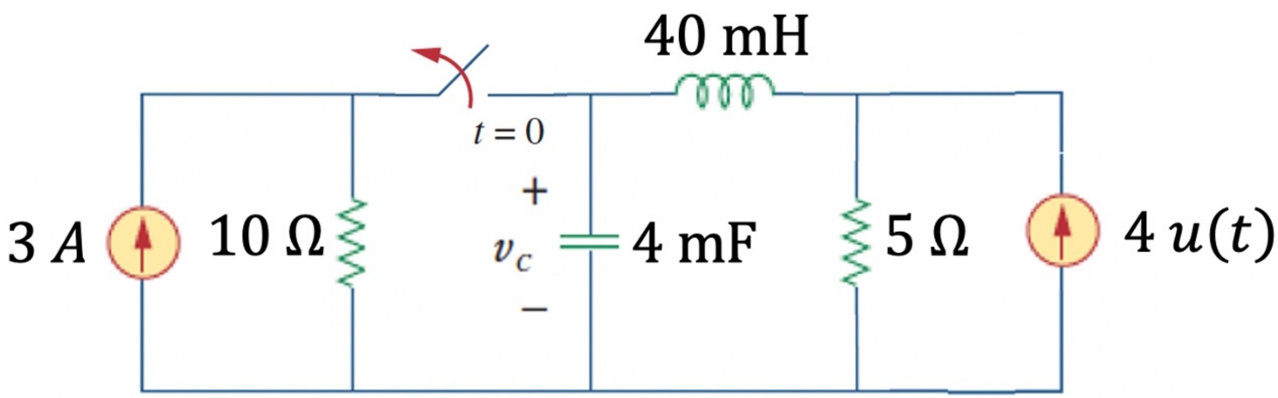
$v_C\,(t) = -85.7\,e^{-16.6t} + 85.7\,e^{-483t}$ V

Practice problem: Find $i_o(t)$



$i_o(t) = 20\, e^{-1000t} - 5\, e^{-4000t}$ mA

Practice problem: Find $v(t)$



$$v(t) = 10\, e^{-62.5t} \cos 48.4t + 2.58.\, e^{-25t} \sin 48.4t + 20\ A$$